A

DISSERTATION

ON THE

CONSTRUCTION AND PROPERTIES

OF

ARCHES.

BY

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PREFACE.

An arch being formed (according to the usual modes of construction) by the apposition of wedges, or sections of a wedge-like form, the properties of arches seem to be naturally derived from those of the wedge, on which principle the inquiries in the ensuing Tract are founded.

By considering the subject on this ground, it appears that the theory of arches may be inferred from geometrical construction, depending only on the known properties of the wedge and other elementary laws of mechanics, without having recourse to the more abstruse branches of geometry in explaining this practical subject, to which a more direct and obvious method of inference seems better adapted.

A geometrical construction for adjusting equilibration on these principles, extended to arches of every form, with the various consequences arising from, or connected with it, are the subject of the ensuing pages, in which rules are investigated, in the first place, for establishing the equilibrium of arches on two distinct conditions, namely, either by adjusting the weights of the sections according to the angles which are contained between their sides, supposed to be given quantities: or, secondly, by supposing the weights of the wedges or sections to be given, and investigating what must be the angles contained by their sides, so that the pressures on them, may be an exact counterpoise to the weight of each section, due regard being had to its place in the arch. In the case when the arch is designed to support an horizontal plane or road, on which heavy weights are to be sustained, the intermediate space between the arch and the horizontal road way, ought to be filled up in such a manner, as not only to afford the support required, but also to add to the strength and security of the entire fabric. If this should be effected by columns erected on the arch, and acting on the several sections by their weight in a direction perpendicular to the horizon, rules are given in the following pages for establishing the equilibrium by adjusting the angles of the sections to their several weights, including the weights of the columns superincumbent, so that the pressure on the sides of each section, may be a counterpoise to its weight, taking into account the place it occupies in the arch. But in structures of this description, the columns of masonry which are erected upon the arches of a bridge, as a support to the road way, cannot be expected to act on the sections of an arch according to the exact proportions required, which are assumed as data in the geometrical propositions, for determining the equilibration, as these proportions would probably be altered either by the differences of specific gravity which may occasionally be found in the materials used, or by differences in the cohesive force, which would prevent the columns from settling and pressing on the several parts of the arch with their full weights, such as the theory requires. Perpendicular columns of iron would not be liable to this objection: by adopting supports of this description, the weights of the columns, added to the weight of the road, would press on the interior arch, to be sustained in equilibrio, by adjusting the angles of the sections to the superincumbent weight, according to the rules determined in the pages which follow. But perhaps the space between the interior arch and the road might be more effectually filled up, by other arches terminated by circular arcs, drawn from centres situated in the vertical line which bisects the entire arch, so as to become united in the highest or middle wedge. The sections of these arches may be adjusted, by the rules here given, so as to become distinct arches of equilibration, which when united, will constitute a single arch of equilibration, similar in form to that which is expressed in the plan of an iron bridge, of one arch, which has been proposed to be erected over the river Thames,* as it is represented in the engraving inserted in the Third Report of the Committee of the House of Commons, for the further Improvement of the Port of London.

According to this plan of construction, each part of the edifice would partake of the properties of equilibration, contributing additional strength and security to the whole building.

In the course of this inquiry, exclusive of the general principles which have been here described, sundry other properties are investigated, which, it is presumed, may be of use in the practice of architecture, in the construction of arches of every kind, as well as in explaining some particulars relating to the subject, which have not hitherto been accounted for in a satisfactory manner.

Some propositions of this kind are comprised in six general rules, inserted in page 19, which are expressed in simple terms, and are easily applicable to practical cases.

Supposing an arch to consist of any number of sections or wedges, adjusted to equilibrium; this arch resting on the two abutments, may be considered analagous to a single wedge, the sides of which are inclined at an angle equal to the inclination of the two abutments: the forces therefore which would be necessary to sustain such an arch or wedge when applied perpendicularly to the sides, ought to be equal to the reaction of the pressures on the two abutments; this principle is found on examination to be verified by referring to the tables annexed;† whether the arch consists of sections, without, or with the load of superincumbent weight, and whether the angles of the sections are equal or

* Designed by Messrs. Telford and Douglass.

† Appendix, Tables, I. II. III. IV. V.

unequal: For according to all these tables, the weight of the semiarch is to the pressure on the corresponding abutment, or the reaction thereof, as the sine of half the angle between the two opposite abutments, is to the radius; which is a proportion equally applicable to the wedge, and to the arch, when adjusted to equilibrium.

From the second of these rules it appears, that the lateral or horizontal pressure of any arch adjusted to equilibrum depends wholly on the weight and angle between the sides of the highest, or middle section: If therefore the weight and angle of this highest section should continue unaltered, the lateral force or pressure will be invariably the same, however the height, the length, the span, and the weight of the whole arch may be varied. This lateral force is called, in technical language, the *drift* or *shoot* of an arch, and the exact determination of it has been considered as a *desideratum* in the practical construction of arches.

When the dimensions of the sections composing the rectilinear or flat arch appeared to follow from the general construction for determining equilibration, the author was inclined to suspect, from the apparent paradox implied in this inference, that some mistake or misapprehension had taken place, either in the general proposition, or in the deductions from it: but finding from trials on a model of an arch of this description, that the sections formed according to the dimensions stated for the flat arch in page 35 of this Tract, Fig. 15, were supported in equilibrio, without any aid from extraneous force, he was convinced that the properties of equilibration deduced from the principle of the wedge, are no less true when applied to practice, than they are in theory. On inquiry it appears, that this species of arch has been long in use among practical artists; the dimensions of the wedges having been formed according to rules established by custom, but without being referred to any certain principle.

A few observations may be here added, concerning the principles assumed in this Tract, as truths to be allowed. It is supposed, that the constituent parts of an arch are portions of wedges, the sides of which are plane surfaces inclined to each other at an angle. Each wedge is considered as a solid body perfectly hard and unelastic, in respect to any force of pressure which can be applied to crush or alter its figure when forming part of an arch, the equilibrium of which is established by the pressure and gravity of the sections only, independently of the ties or holdings, which are applied for the purpose of preventing the extrusion of the wedges by the force of any occasional weight which may be brought to press on the arch. These fastenings supply in some degree the place of the natural force, by which the parts of solid bodies cohere till they are separated by artificial means. When the weights of the sections are not very great, a defect of equilibrium to a certain extent may subsist, without producing any material change in the figure of the arch, or endangering the security of the fabric. But if heavy massive blocks of stone or iron should be placed together in the form of an arch, without being well adjusted, any considerable defect of equilibrium would cause a stress on the fastenings, which would overbear the weak alliance of cement or the mechanical ties and fastenings, that are applied to prevent the separation of the sections. In other cases, when sections of less weight are used, the cohesion which takes place between the surfaces of blocks of stone, with the aid of cement, and fastenings of various kinds, may impart a considerable degree of strength to edifices; insomuch, that although many arches have been counterpoised according to rules which produce rather a fortuitous arrangement of materials for forming the equilibrium, than an adjustment of it, according to correct principles, the cohesion of the parts have, notwithstanding, preserved them from falling; or from experiencing any considerable change of form. But this power has been sometimes too much relied on, especially by the architects of the twelfth, thirteenth, and the centuries immediately succeeding, who, in particular instances, entertained the bold idea of erecting lofty pillars, subject to a great lateral pressure, without applying any counterpoise. The consequence has been a distortion of figure; too evidently discernible in the pillars which support the domes in most of the old cathedral churches. However great the force of cohesion may be, which connects the parts of buildings, every edifice would be more secure by having all the parts of it duly balanced, independently of cohesion or mechanical fastenings, by which means, that distortion of shape would be prevented which the want of equilibrium in structures, must always have a tendency to create, whether the effects of it should be sufficient to produce a visible change of figure, or should be too small to be discernible by the eye. When arches are not perfectly balanced, and a change of figure ensues, the only security for the preservation of the fabric from entire disunion, is the excess of the cohesive force above the force tending to separate the parts of the building, arising from the want of counterpoise; and as cohesion is a species of force, which cannot be estimated with exactness, where the circumstances of an edifice are such as may weaken this force, or render the effects of it precarious, the more attention is necessary to establish a perfect equipoise between the weights and pressures of the several parts.

A

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As the exterior termination of an arch always exceeds the interior curve (usually called the curve of the arch), the sections or wedges of which it is composed will partake of a similar disproportion, the length of the exterior boundary in each wedge always exceeding that of the interior. A consequence of this wedge-like form is, that the weight of each section by which it endeavours to descend towards the earth, is opposed by the pressure the sides of it sustain from the sections which are adjacent to it. If the pressure should be too small, the wedge will not be supported, but will descend with greater or less obliquity to the horizon, according to its place in the arch. If the pressure should be too great, it will more than counterpoise the weight of the section, and will force it upward. The equilibrium of the entire arch will consequently depend on the exact adjustment of the weight of each section or wedge, to the pressure it sustains, and the angular distance from the vertex. This equilibrium is understood to be established by the mutual pressure and gravity of the sections only, independent of any aid from friction, cohesive cement, or fastenings of any kind.

When an arch is erected, fastenings are necessarily applied, to prevent the extrusion of the wedges by any force of weight which may be occasionally brought to press upon the several parts of the arch. The ensuing geometrical constructions are intended, in the first instance, to adjust the equipoise of the wedges or sections which are disposed according to the direction of any curve line that may be conceived to pass through the extremities of their bases, requiring only, as conditions to be given, the weight of the highest or middle wedge, and the angles contained by the sides of the several wedges or sections. From these *data*, the weight of each section is to be inferred, so that when the whole are put together, they may balance each other in perfect equipoise in every part. Since, according to this construction, the weights of the sections are dependent on the given angles between their sides, the exterior termination of the sections, or the weights superincumbent on them, will usually take some form which cannot be altered, without a change in the conditions given: to effect this change, when requisite, other considerations will be necessary, which are the subject of the latter part of this tract.

For these reasons it appears, that the principle of establishing the equilibrium of an arch, by inferring the weights of the sections from their angles, not requiring a determinate form to the exterior boundary, is best suited to the construction of those arches which are erected either to connect the several parts of an edifice, or for their support and ornament, to which they so eminently contribute in many of those ancient monuments of skill and magnificence, which still remain to be contemplated with delight and admiration.

According to this principle of construction, the architect is not restrained to curves, observing any particular law of curvature, but may adopt any that require mechanical delineation only, for describing the forms of arches he may wish to erect; to this advantage is added, that of having each arch balanced within itself, and the pressure on each part, as well as on the abutments, exactly ascertained. And, as the angles of the sections may be varied at discretion, by properly altering the adjustment of the equilibrium, according to the rules here given, the force of pressure on the abutments may be made to take any direction which best contributes to the strength of the edifice, so far as the limits of those rules will allow.

In addition to the principles which have been the subject of the preceeding observations, the following case is next to be considered. It has been already observed, that when the weights of the sections are inferred from the angles between their sides, the heights of the masses added to the sections, or making a part of them, will be terminated by some line depending on the dimensions of these angles. But arches are often constructed for the purpose of supporting considerable weights, the terminations of which are required to be of particular given forms. In the case of bridges, a wall of masonry is usually erected on the arches, as a support to the road-way, which is always horizontal, or nearly so; the superincumbent weight of this wall, by adding proportionally to the weights of the sections, must require a stronger force of pressure on their sides as a counterpoise to it; this is effected in consequence of two alterations, by which the loaded arc differs from the arc constructed only for supporting its own weight. 1st. The weight of the highest or middle section, being augmented, increases the pressure, and the reaction between the surfaces of all the sections; and to compensate for the different degrees of weight which are superadded to the other sections, the angles contained between their sides, and the pressures upon them, are to be increased in due proportion. For as the angles of the wedges are increased, a given force of pressure acting on their sides, will have the greater effect in supporting the intermediate wedges. The heights of the wall built on the several sections proportional to the weights superincumbent on them, are supposed to be given quantities, so that the upper extremities may terminate in an horizontal, or any other given line: instead therefore of inferring those heights, or weights, from the angles of the sections considered as given, according to the principle of construction, which has been described in the preceding pages, we are to consider the heights of the wall, or weights on the several sections, as given quantities, and to infer from them, what must be the magnitudes of the angles contained by the sides of the sections, so that the weight of each, including the weight superincumbent, may be an exact counterpoise to the pressure on the sides of it. The curve line, which passes through the bases, may be of any form, without affecting the equilibrium established according to these principles. For the counterpoise of gravity and pressure between two or more wedges, is wholly independent of the line which may be drawn through their bases.

If it should be objected that the more nearly an arch approaches to a right line, the less weight it will securely bear, it may be replied, that this insecurity is caused by circumstances which are quite independent of the equilibrium.

If the materials of which an arch is constructed, were perfectly hard and rigid, so as not to be liable to the smallest change in their form, and the abutments were immoveably fixed, an arch, when the sections have been adjusted, although but little deviating from a right line, would be equally secure, in respect to equilibrium, with a semi-circular, or any other arch.

From these general observations, the object of the ensuing tract appears to consist principally in the solutions of two statical problems, which may be briefly expressed in the following terms; 1st, from having given the angles contained by the sides of the wedges which form an arch, together with the weight of the highest or middle section, to infer the weights of the other sections; and conversely, from the weights of each wedge given, together with the angle of the first section, to determine the angles between the sides of the other sections, so as to form an arch perfectly balanced in all its parts.

In the construction of circular arches, the joinings of the sections, or sides of the wedges, are usually directed to the centre of the circle. In the following constructions, the sides of the wedges are directed to any different points; but there is no reason to suppose, that the equilibrium of the arch would be altered, or that the construction would be less secure, from this circumstance.

Considering that an arch supports the weights which press upon it, and preserves its form in consequence of the wedge-like figures of the sections; the principle of its construction and properties seem naturally to be referred to those of the wedge, which principle has been adopted in the ensuing disquisition founded on plain geometry and statics, or the doctrine of equilibrium or equipoise, as established by Gallileo and Newton.

Fig. 1. K C G A, D B G A, D B F E, represent three of the

sections or wedges which form an arch, the lower curve of which passes through the points C, A, B, E, &c.

The wedges are also, for brevity, denoted by the letters B, A, and C respectively.

The highest wedge of the arch is G A D B, which (being here considered isosceles) is terminated on each side by the lines D B, G A inclined to each other at the angle B O A, which is termed, for the sake of distinction, the angle of the wedge or section. The termination of this wedge on the lower side is the line B A, the extremities of which coincide with the curve of the arch, and on the upper part, by the horizontal line D G parallel to B A. If therefore D G is bisected in the point V, a line V O joining the points V and O, will be perpendicular to the horizon. In like manner, the inclination of the sides K C, G A forms the angle of the wedge C, and the inclination of the two sides D B, F E is the angle of the wedge B. In the construction of arches, the angles of the sections are commonly made equal to each other, but in a general investigation of the subject, it will be expedient to consider the angles of the sections of any magnitude, in general, either as quantities given for forming the equilibrium of the arch, by the adjustment of their weights, or as quantities to be inferred, from having the weights of their sections given.

Fig. 1. The wedge A when unimpeded, endeavours by its gravity, to descend in the direction of the line V O, but is prevented from falling by the pressure of the wedges on each side of it, acting in the direction of the lines P Q, K I, perpendicular to the surfaces D B, G A respectively.

By the principles of statics, it is known, that if the force P Q, or its equal K I, should be to half the weight of the wedge, in the same proportion which the line O D bears to V D, that is, in the proportion of radius to the sine of half the angle of the wedge V O D, the weight of the wedge will be exactly counterpoised by these forces; and conversely, if any wedge is sustained in equilibrio by forces applied perpendicularly to the sides, these forces must be to the weight of the wedge in the proportion which has been stated.

It is to be observed, that all pressures are estimated in a direction perpendicular to the surfaces impressed; for if the direction of the pressure should be oblique, it may be resolved into a force perpendicular to the surface, and some other force, which neither increases nor diminishes the pressure.

Fig. 2. If the wedge A when unsupported, should not be at liberty to descend freely in the direction of the vertical line V O, but should be moveable only along the line G A, considered as a fixed abutment,* the force P Q singly applied will sustain it in equilibrio, the reaction of the abutment supplying the necessary counterpoise. For produce P Q (Fig. 2.) till it intersects the line G A in the point X, and in line X P, take M X equal to P Q; the force P Q, considered as applied perpendicular to the surface D B, will have no effect in impelling the wedge toward the point O in the direction D B, or in the opposite direction B D, but the same force M X acting in an oblique direction on the line GA, may be resolved into two forces, M R perpendicular to G A, acting as pressure on it, and the force R X which impells the wedge directly from the point O in the direction of

* To prevent repetitions and unnecessary references, it is to be observed in the following pages, that the lower surface of each section is considered as a fixed abutment, on which the weight of the section, and of all the sections above it, are supported. For this reason, the angle between the lower surface of any section and the vertical line, is termed, for brevity, the angle of the abutment of that section.

the line A G. Through any point A, in the line A G, draw A *a* parallel to the line V O representing in quantity and direction the weight of the wedge A; through the point *a* draw *a* H perpendicular to A G; then will H A represent the force by which the wedge endeavours to descend in the direction of the line G A, considered as a fixed abutment. If then the line H A should be proved equal to the line R X, the contrary directions of these equal forces will balance each other, and the wedge so impelled will remain at rest in equilibrio. The proof that the lines H A and X R are equal is as follows.

By the construction, the line A *a* represents the weight of the wedge A, and the angle H A *a* is equal to the angle V O G, or half the angle of the wedge. And because the line M X is perpendicular to B D, and M R perpendicular to G A, it follows that the inclination of the lines G A, D B is equal to the inclination of the lines M X and M R, or the angle X M R is equal to the angle D O G. By the construction, and the properties of the wedge

$$MX : \tfrac{1}{2} Aa :: \text{radius} : \text{sine of } VOG$$

and

$$RX : MX :: \text{sine } XMR : \text{radius}$$

Joining these ratios

$$RX : \tfrac{1}{2} Aa :: \text{sin. of } XMR : \text{sin. } VOG$$

or because X M R is equal to D O G

$$RX : Aa :: \text{sin. } DOG : 2 \text{ sin. } VOG$$

also by construction $\left.\right\}$ $Aa : HA :: \text{radius} : \text{cos. } HAa \text{ or } VOG$

Joining these ratios

$$RX : HA : \text{radius} \times \text{sin. } DOG :: 2 \times \text{sin. } VOG \times \text{cos. } VOG.$$

But because the angle D O G is double to the angle V O G, from the principles of trigonometry it follows, that sin. D O G

$\times$ radius is equal to 2 sin. V O G $\times$ cos. V O G: therefore since the line R X is to the line A H, as sin. D O G $\times$ radius, to 2 $\times$ sin. G O V $\times$ cos. G O V, it follows that the line R X is equal to the line A H.

All the successive sections or wedges which form the arch being by the supposition balanced and sustained by their gravity and mutual pressure, independent of any other force, if the whole of the arch is considered as completed, whatever force P Q (Fig. 1 and 2.), is necessary for sustaining the wedge A G D B in equilibrio will be supplied by the reaction of the wedge adjacent to the surface D B. And in every part of the arch, when perfectly balanced, whatever force of pressure is communicated to any section from that which is contiguous to it, the force of reaction will be precisely equal between the two sections. It appears, therefore, that the equilibrium of the wedges will depend on the due adjustment of their successive weights to the pressures sustained by the sides of the sections. To effect this, the several sections are to be successively balanced; first, the wedge A alone; considering it as moveable along the line G A, as a fixed abutment.

It has been shewn, that if the force (Fig. 1.) P Q is to half the weight of the wedge A, as the line D O is to the line V D, this force P Q acting perpendicularly against the surface D B, and communicating an equal pressure M X obliquely on the line G A will sustain the wedge A from descending along the line G A. In the next place, let the wedge B (Fig. 2.) be adjusted to equipoise conjointly with the wedge A; and let it be required to ascertain the weight of B, in proportion to the weight of A, so that they may continue in equilibrio, when moveable along the fixed abutment K B. (Fig. 2 and Fig. 3.) To effect this, produce M R till it intersects the line K B in the point V, and in the line R M produced, take M N equal to the line H *a*. In V N take V Q equal to R N; and from the point Q draw Q W perpendicular to the line K B produced if necessary, intersecting it in the point W: from any point B in the line K W, take B H equal to W V, and through the point H draw H *z* (indefinite) perpendicular to the line K B. Through B draw B *x* (indefinite) parallel to the line V L O, intersecting the line H *z* in the point *b*; then, the line B *b* will represent by construction the weight of the section B, when the line A *a* denotes the weight of the section A, and the wedges are balanced in equilibrio, although freely moveable in the directions of the lines G A, K B.

For the pressure P Q or M X, which impels the wedge A upward along the line A G with the force R X, is perfectly counterpoised by the force of gravity A H referred to that direction, because it has been proved, that the line R X is equal to A H. If the pressure H *a* or M N* on the line G A, arising from the weight of the section A, be added to the pressure M R, the sum or R N, will be the entire pressure on the surface A G, equal to V Q by construction, or the oblique pressure on the line B K; that is $RM + Ha = RN = QV$; Q V is resolved into two forces, Q W, perpendicular to K B produced, and W V in the direction of that line. The force Q W, acts as pressure on the line B K, and the force W V impels the wedge in a direction contrary to gravity along the line B K. The weight of the section B, is by the construction denoted by the line B *b*, and being resolved into two forces, H B acting in the direction of that line, and H *b* perpendicular to it; the force H *b* acts as pressure, and the force H B is that part of the weight which impels the wedge B to descend in the direction of the line K B. But, by the construction, the line H B, is equal to the line W V,

* The lines H *a* M N, represent the quantity and direction of these pressures, but are not to be understood as determining the point or place where the pressures are applied.

these equal and contrary forces will therefore balance each other; and the wedge so impelled, will remain at rest, so far as regards the direction of the line B K. In respect to the forces Q W, H *b*, which act in a direction perpendicular to the surface K B, they are perfectly balanced by the reaction of the abutment B K, or the reaction of the wedge C, when the forces in the direction of the line F C have been adjusted to equilibrium.

The result is, that when the weights of the sections A and B, have been adjusted according to the preceding construction, each of the forces both of pressure and gravity is exactly counteracted by an opposite force which is equal to it. (Fig. 3.) The weights of the wedges C and D are adjusted to equilibrium by a similar construction. Produce Q W till it intersects F Z in the point X. In W Q produced, set off Q *b* equal to H *b*, and in the line X *b* take X Y equal to W *b*; through the point Y draw Y Z perpendicular to F C produced, and from any point C in the line C F, take C H equal to Z X; through H draw the indefinite line H *z* perpendicular to C F, and through the point C draw the indefinite line C *x* parallel to V L O; the intercepted line C *c* will represent the weight of the section C. The weight of the section D is constructed on the same principle, by making the line D H equal to the line F B, and drawing D *x* parallel to V O, and H *z* perpendicular to I D, cutting off the line D *d*, which is the weight of the section D. If the sections D, C, B, A, then adjusted, are placed contiguous, and the opposite semiarch is completed, when the extreme sections D, D, are supported on the two abutments, the whole will remain in equilibrio, although freely moveable in the direction of the lines I D, F C, K B, G A, D C, &c.

These and the remaining weights having been thus adjusted, so as to form an equilibrium, the lines A *a*, B *b*, C *c*, D *d*, to which they are proportional, might be determined by lineal construction, according to the rules which have been given; but as the correctness of such graphical delineations depends both on the excellence of the instruments employed, and on the skill of the person who uses them, to supply the want of these advantages in any case that may occur, as well as to view the subject under a different form, it may be expedient to express the several weights and pressures which have been hitherto represented geometrically, by analytical and numerical values. From the preceding observations, it has appeared, that the weights and pressures depend in a material degree, on the weight of the highest or middle wedge A, which is bissected by the vertical line V O. (Fig. 1, 2, and 3.) The weight of this section has been denoted by the line Aa in the construction, and is represented in the analytical values, by the letter w; all other weights being in proportion to it. The initial pressure acting obliquely against the side of the wedge A G, represented by the line P Q $=$ M X, has been found $= \frac{w}{2 \sin. VOD} = p$; and because the line M X is perpendicular to D C, and M R is perpendicular to G A, it follows that the inclination of the lines D C, G A, is equal to the angle X M R; or if A° is put to represent the angle contained between the sides of the wedge A, it will follow that A° $=$ X M R $=$ A O B.

Since therefore $p = \frac{w}{2 \times \sin. \frac{1}{2} A^\circ} =$ MX $=$ PQ, the direct pressure against the surface G A, that is, M R is $=$ M X $\times$ cos. A° $= p \times \cos. A^\circ$. And as the additional pressure arising from the weight of the wedge A is $=$ Ha $=$ A$a \times \sin. \frac{1}{2} A^\circ = w \times \sin. \frac{1}{2} A^\circ$, the entire pressure on the surface G A $= p \times \cos. A^\circ + w \times \sin. \frac{1}{2} A^\circ$; or since $p = \frac{w}{2 \times \sin. \frac{1}{2} A^\circ}$, the entire pressure on G A $=$ M R $+$ Ha $= \frac{w \times \cos. A^\circ.}{2 \times \sin. \frac{1}{2} A^\circ} + w \times \sin. \frac{1}{2} A^\circ = \frac{\cos. A^\circ + 2 \sin.^2 \frac{1}{2} A^\circ}{2 \sin. \frac{1}{2} A^\circ} \times w$; or because $2 \sin.^2 \frac{1}{2} A^\circ =$ the versed sine of the angle A°, it follows that

$MR + Ha = VQ = \frac{w}{2 \times \sin. \frac{1}{2} A^\circ} = p$. But, because the line QV is perpendicular to GA, and QW is perpendicular to KB, the inclination of the sides KB, GA of the wedge B is equal to the angle VQW, which may be denoted by B°; and because QV $= p$, it follows that QW $= p \times \cos. B^\circ$, and VW $= p \times \sin. B^\circ$; which is equal to the line HB, by the construction. If, therefore, the angle HBb, or the angle at which the abutment KB is inclined to the vertical line VO, or Bb should be represented by V^b, since VW or BH $= p \times \sin. B^\circ$, Hb will be $= p \times \sin. B^\circ \times \text{tang.} V^b$, and the line B$b$ will be $p \times \sin. B^\circ \times \text{secant } V^b \times$ which is the weight of the section B; the pressure on the next section, or C, is QW + Hb, which is $= p \times \cos. B^\circ + p \times \sin. B^\circ \times \text{tang.} V^b$: let this be made $= q$; then the weight Cc of the section C will be found in like manner to be $= q \times \sin. C^\circ \times \sec. V^c$, and the pressure on the next section D $= q \times \cos. C^\circ + q \times \sin. C^\circ \times \text{tang.} V^c$, and so on, according to the order of weights and pressures which are here subjoined.

It is to be observed, that A°, B°, C°, D°, &c. denote the angles of the sections A, B, C, &c. V^a signifies the angle of inclination, to the vertical, of the line GA, on which the section A rests, = to the angle GOV or HAa. In like manner V^b is the inclination to the vertical of the line KB, on which the section B rests = to the angle HBb: V^c is the inclination to the vertical of the line FC, on which the section C rests, = to the angle HCc, and so on. The initial pressure $p = \frac{w}{2 \times \sin. \frac{1}{2} A^\circ}$.

Sections.	Weights of the Sections.	Pressures on the Sections next following.
A	w	$p \times \cos. A^\circ + p \times \sin. A^\circ \times \text{tang.} V^a = p$
B	$p \times \sin. B^\circ \times \sec. V^b$	$p \times \cos. B^\circ + p \times \sin. B^\circ \times \text{tang.} V^b = q$
C	$q \times \sin. C^\circ \times \sec. V^c$	$q \times \cos. C^\circ + q \times \sin. C^\circ \times \text{tang.} V^c = r$
D	$r \times \sin. D^\circ \times \sec. V^d$	$r \times \cos. D^\circ + r \times \sin. D^\circ \times \text{tang.} V^d = s$
E	$s \times \sin. E^\circ \times \sec. V^e$	$s \times \cos. E^\circ + s \times \sin. E^\circ \times \text{tang.} V$

Or the weights of the sections and pressures on them may be expressed somewhat differently thus: the weights of the successive sections being denoted by the letters a, b, c, d, respectively.

Sections.	Weight of the Sections.	Pressures on the Section next following.
A	$a = w$	$p \times \cos. A^{\circ} + a \times \sin. V^{a} = p$
B	$b = p \times \sin. B^{\circ} \times \sec. V^{b}$	$p \times \cos. B^{\circ} + b \times \sin. V^{b} = q$
C	$c = q \times \sin. C^{\circ} \times \sec. V^{c}$	$q \times \cos. C^{\circ} + c \times \sin. V^{c} = r$
D	$d = r \times \sin. D^{\circ} \times \sec. V^{d}$	$r \times \cos. D^{\circ} + d \times \sin. V^{d} = s$
E	$e = s \times \sin E^{\circ} \times \sec. V^{e}$	$s \times \cos. E^{\circ} + e \times \sin. V^{e} = t$

In the following illustration of these analytical values, the angles of the sections are assumed equal to 5° each, and the weight of the first section is put $= 1$: consequently, the initial pressure or $p = \frac{1}{2 \times \sin. 2^{\circ} 30'} = 11.46279 =$ the pressure on the second section or B: for inferring the weight of the second section, we have $p = 11.46279$, $\sin. B^{\circ} = .087156$, $V^{b} = 7^{\circ} 30'$, and $\sec. V^{b} = 1.008629$; wherefore the weight of B $= p \times \sin. B^{\circ} \times \sec. V^{b} = 1.00767$. The pressure on the section next following, or C, $= p \times \cos. B^{\circ} + p \times \sin. B^{\circ} \times \text{tang. } V^{b} = 11.550708 = q$, and therefore the weight of C $= q \times \sin. C^{\circ} \times \sec. V^{c} = 1.03115$, and the pressure on the next section D $= q \times \cos. C^{\circ} + q \times \sin. C^{\circ} \times \text{tang. } V^{c} = 11.72992$; and thenceforward, according to the values entered in the table No. 1.

When any number of sections have been adjusted to their proper weights on each side of the vertical line V O, the whole being supported on the abutments, on the opposite sides, will remain in equilibrio, balanced by the mutual pressure and gravity of the sections only; so that if the contiguous surfaces were made smooth, and oil should be interposed between them, none of the sections would be moved from their respective places.

But if the wedges should be put together without adjustment, the weight of the sections near the abutments, if too great, would raise those which are nearer the vertex, and would themselves descend from their places, in consequence of their over-balance of weight; contrary effects would follow, from giving too small a weight to the lower sections. In either case, they would not retain their places in the arch, unless the fastenings applied in order to prevent them from separating, should be stronger than the force created by the imperfect equilibrium, impelling the sections toward a position different from that which they ought to occupy in the arch.

But a distinction is to be made between the deficiency of equilibrium which is inherent in the original construction, and that which is caused by an excess of weight, which may occasionally be brought to press on an arch.

In the case of occasional weight, such as loads of heavy materials which pass over the arches of a bridge, the stress on the joinings is temporary only; whereas the stress which arises from a want of equilibrium in the construction, acts without intermission, and in a course of time, may produce disturbance in the fastenings, and in the form of the arch itself, which might resist, without injury, a much greater force that acts on the several parts of it during a small interval of time.

When the wedges which form an arch have been adjusted to equilibrium, the whole will be terminated at the extremities by two planes coinciding with the abutments, and the entire arch will in this respect, be similar in form to a single wedge, the sides of which are inclined to each other, at an angle equal to the angle at which the planes of the abutments are inclined. (Fig. 4.) I V C, represents an arch adjusted to equilibrium, and terminated by the curve lines I V C, F B D.

The extremities of the arch are placed on the abutments I F, C D, which lines when produced, meet in a point O. If the arch be bisected in V, a line joining the points V and O will be perpendicular to the horizon. Let the line K L be drawn perpendicular to the surface I F and M N W perpendicular to C D. The arch acts by its pressure in a direction perpendicular to the abutments, the reaction of which is equal and contrary to the pressure, and in the direction of the lines K L, M N.

From the principles of statics before referred to, it appears, that if forces are applied in the directions K L, M N, perpendicular to the surfaces I F, C D, considered as the sides of a wedge, those forces K L, M N, will sustain the wedge, provided each of them should be to the weight of the semiarch, as radius is to the sine of the semiangle of the wedge, that is as radius to the sine of V O C. If therefore, the abutment should be considered as removed, and two forces or pressures equal to the forces K L, M N, should be substituted instead of them, each being in the proportion that has been stated, the wedge or arch will be sustained in the same manner as it was by the abutments.

The force that sustains the arch or wedge, is the reaction of the abutments, which is exactly equal and contrary to the force of pressure upon either of them, and has been ascertained in the preceding pages, when an arch consisting of any number of sections is adjusted to equilibrium as in the table, No. 1. If, therefore, the forces K L, or M N, be made equal to the pressure on the abutment, determined as above, the following proportion will result:

As the force of pressure on the abutment is to half the weight of the wedge or arch V B C D, so is radius to the sine of the inclination of the abutment to the vertical line, or is as radius to the sine of the angle V O C.

Thus, referring to the order of weights and pressures, stated in page 13, let the semi-arch consist of any number, suppose five sections, including half the first section, or $\frac{1}{2}$ A: then the weight of the semi-arch will be $\frac{1}{2} A + B + C + D + E$.

The pressure on the abutment which sustains the section E is $= s \times \cos. E^{\circ} + s \times \sin E^{\circ} \times \text{tang.} V^{e} = t$. E° is the angle of E, the fifth wedge $= 5^{\circ}$, and V^{e} is the angle of inclination to the vertical of the abutment on which the section E rests: or $V^{e} = 22^{\circ}. 40'$; then according to the proportion which has been stated, as the sum $\frac{1}{2} A + B + C + D + E$ is to the pressure t, so is the sine of the angle V^{e} to 1, which may be verified by referring to the numerical Table I. according to which, $\frac{1}{2} A + B + C + D + E = 4.7436$; $V^{e} = VON = 22^{\circ}\ 30'$; and the pressure on this abutment appears, by the same table, to be $= 12.3954$: the sine of $22^{\circ}\ 30' = .38268$, which being multiplied by the pressure 12.3954, the product is 4.7434, scarcely differing from 4.7436, as entered in the table. In general, let the letter S denote half the weight of an arch, when adjusted to equilibrium: and let Z represent the pressure on the abutment, the inclination of which to the vertical is V^{z}; then $S = Z \times \sin. V^{z}$.

Fig. 4. I F B D C represents an arch of equilibration, which is bisected by the vertical line VO; CD is one of the abutments inclined to the vertical line in the angle VOC: through any point N of the abutment draw the line MW perpendicular to CD, and through N draw the line TQ perpendicular to the horizon. In the line N W set off N E representing the pressure on the abutment CD, and resolve NE into two forces, EA, in the direction parallel to the horizon, and AN perpendicular to it. Then, because the angle ANF is equal to the angle NOV, or the inclination of the abutment to the vertical, denoted by the angle V^{z}, it follows, that the angle NEA is equal to the angle ANF or VOC $= V^{z}$, from

whence the same proportion is derived, which has been otherwise demonstrated before, namely, as N E : N A :: 1 : sin. VOC; that is, the force of pressure on the abutment is to the weight of the semiarch, as radius is to the sine of V^z = N O V. It is observable that the additional weight of wedges, by which the weight of the semiarch is increased, reckoning from the vertex, or highest wedge, always acts in a direction perpendicular to the horizon; but neither increases nor diminishes the horizontal force, which must therefore remain invariably the same, and is represented by the line EA. But the initial force of pressure which has been denoted by the letter p, is not precisely horizontal, being in a direction perpendicular to the surface of the first wedge A, although it is very nearly parallel to the horizon, when the angle of the first section is small, and might be assumed for it as an approximate value: a force which is to the initial pressure, or $p = \frac{w}{2 \times \text{sin.} \frac{1}{2} A^\circ}$, as the cosine* of $\frac{1}{2} A^\circ$ (Fig. 5.) to radius, will approximate still more nearly to the constant force, the direction of which is parallel to the horizon, and is therefore $= \frac{w \times \text{cos.} \frac{1}{2} A^\circ}{2 \times \text{sin.} \frac{1}{2} A^\circ} = \frac{w}{2 \times \text{tang.} \frac{1}{2} A^\circ}$. For the sake of distinction, let this force be represented by $p' = \frac{w}{2 \times \text{tang.} \frac{1}{2} A^\circ}$, to be assumed as an approximate value, which will differ very little from the truth when the angles of the sections are small, and in any case will be sufficiently exact for practical purposes. Then, because E A is denoted by p', N E = Z (Fig. 4.) represents the pressure on the abutment, and N E is the secant of the angle N E A, or V O C, to radius E A, it appears that $Z = p' \times$ secant V^z: by the same construction, A N is the tangent of the angle N E A to radius E A; also A N is the sine of the angle N E A, or V O C, to the radius N E.

* Note in the Appendix.

The following general rules are derived from the proportions, which have been inferred in the preceeding pages:

Rule i. The initial pressure is to the weight of the first section, including the weight superincumbent on it, as radius is to twice the sine of the semiangle of the middle, or highest wedge, or $p = \frac{w}{2 \times \text{sin.} \frac{1}{2} A^{\circ}}$.

Rule ii. The horizontal force, which is nearly the same in every part of the arch, is to the weight of the first section, as radius is to twice the tangent of the semiangle of the first section, or $p' = \frac{w}{2 \times \text{tang.} \frac{1}{2} A^{\circ}}$.

Rule iii. The horizontal or lateral force is to the pressure on the abutment, as radius is to the secant of the inclination of the abutment to the vertical, or $Z = p' \times \text{sec. } V^{x}$.

Rule iv. The horizontal force is to the weight of half the arch as radius is to the tangent of the inclination of the abutment to the vertical, or $S = p' \times \text{tang. } V^{x}$.

Rule v. The weight of the semiarch is to the pressure on the abutment, as the sine of the said inclination of the abutment is to radius, or $S = Z \times \text{sin. } V^{x}$.

Rule vi. The horizontal force is to the pressure on the abutment as the cosine of the inclination of the abutment is to radius, or $p' = Z \times \text{cos. } V^{x}$.

By these rules, the principal properties of the arch of equilibration are expressed in simple terms, and are easily applicable to practical cases.

Rule 3d. The horizontal force, or p', being the weight divided by twice the tangent of the semiangle of the first section, determines the pressure on any abutment of which the inclination to the vertical line is V^{x}; the pressure being $= p' \times \text{secant } V^{x}$.

D 2

Rule 4th. The weight of the semiarch, when adjusted to equilibrium, is found by the fourth rule to be $= p' \times$ tang. V^2; or the horizontal pressure increased, or diminished, in the proportion of the tangent of the vertical distance of the abutment to radius. From this property, the reason is evident, which causes so great an augmentation in the weights of the sections, when the semiarch, adjusted to equilibrium, approaches nearly to a quadrant, and which prevents the possibility of effecting this adjustment by direct weight, when the entire arch is a semicircle.

Rule 5th. The fifth rule exemplifies the analogy between the entire arch when adjusted to equilibrium, and the wedge. For let the angle between the abutments be made equal to the angle of the wedge, the weight of which is equal to the weight of the arch; and let Z be either of the equal forces, which being applied perpendicular to the sides of the wedge, sustain it in equilibrio: then by the properties of the wedge, the force Z is to half the weight of the wedge as radius is to the sine of the semiangle of the wedge, which is precisely the property of the arch; substituting the angle between the abutments instead of the angle of the wedge, and the pressure on either abutment instead of the force Z.

Rule 6th. The lateral pressure, or the pressure on the abutment, reduced to an horizontal direction, is nearly the same in all parts of the arc; being to the weight of the first section, as radius is to twice the tangent of the semiangle of the wedge.

The force of pressure on the abutment is therefore at every point resolvable into two forces; one of which is perpendicular to the horizon, and is equal to the weight of the semiarch; and the other is a horizontal or lateral force, which is to the weight of the first section, as radius is to twice the tangent of the semiangle of that section.

These conclusions are the more remarkable from the analogy they bear to the properties of the catenary curve, although they have been deduced from the nature of the wedge, and the principles, of statics only, and without reference to the catenary or other curve, and will be equally true, when applied to the sections of an arch, which are disposed in the form of any curve whatever.

Many cases occur, in the practice of architecture, in which it must be highly useful to form an exact estimate of the magnitude, and direction of pressure, from superincumbent weight, both on account of the danger to be apprehended if such pressure is suffered to act against the parts of an edifice without a suitable counterpoise; and from the consideration, that when the extent of the evil to be provided against is not certainly known, it is probable, that more labour and expense will be employed in making every thing secure, than would have appeared necessary if the pressure to be opposed had been exactly estimated.

The Gothic cathedrals, and other edifices built in a similar style of architecture, which very generally prevailed in this and other countries of Europe, during several centuries, have been constructed on principles to which the preceding observations are not intirely inapplicable.

The striking effects for which these structures are remarkable, seem principally to be derived from the loftiness of the pillars, and arches springing from considerable heights, which could not be securely counterpoised without making great sacrifices of external appearance, and bestowing much labour and expense in imparting sufficient stability to the whole fabric: while, therefore, the eye is gratified by the just proportions and symmetry of design, displayed by the interior of these edifices, with an apparent lightness of structure, scarcely to be thought compatable with the use of such materials, the external building presents the unpleasing contrast of angular buttresses, with their massive weights, which are indispensably necessary for the preservation of the walls, as a counterpoise to the lateral pressure from the ponderous arched roofs.

That the security provided has been perfectly effectual, is evident from the solidity and duration of these buildings; but whether any part of their heavy supports might have been spared, or whether the form of applying them might not have been changed, without diminishing the security of the walls, is a question which would require much practical experience and information to decide. An estimate of the lateral force from arches of this description, may be readily obtained by referring to the rules given in the preceding pages; from which it appears, that the lateral or horizontal force arising from the pressure of any arch is always to the weight of the highest or middle wedge, as radius is to 2 × tangent of the semiangle of the wedge. Some of the highest wedges, in roofs of this description, are said to weigh two ton; the angle of the wedge may be taken (merely to establish a case for illustrating the subject) equal to $3°$: the tangent of half this angle, or $1° 30'$, is .02618, and the lateral force, or pressure from any part of the arch, will be $\frac{2}{2 \times .02618} = 38.2$ ton, which weight of pressure, acting on walls of great height, must certainly require a very substantial counterpoise for their support; and it is for this purpose, that a buttress is erected against the external walls corresponding to the key-stone of each arch. In this estimate, the arch is supposed to have reference to one plane only; which passes through both the buttresses and the key-stone: but in the case of groined arches, or such as are traversed by other arches crossing them diagonally, on which the same key-stone acts, the lateral pressure in any one plane will be less than has been found according to the preceding estimate.

In the preceding geometrical constructions, the angles of the several sections have been assumed equal to each other, or as given, although of different magnitudes, from which data the weights of the sections have been inferred when they form an arch of equilibration.

Let D C V D, &c. (Fig. 6.) be the curve-line passing through the bases of the sections which form an arch: because the wedges increase as they approach the abutment; the exterior line *a b c d*, &c. will take a form not very dissimilar to that which is represented in the figure, (Fig. 6.): the arch being here adjusted to equilibrium in itself without reference to any extraneous weight or pressure. Although in many cases, it is immaterial what may be the form of the line *a b c*, &c. yet it often happens, that the termination of the sections, or exterior boundary, must of necessity deviate greatly from that which is represented in the figure; particularly in the case of bridges, over which a passage or road-way is required to be made, which is either horizontal, or nearly so. Let E D L D E (Fig. 7.) represent an arch by which an horizontal road, P *a* Q, is supported. For this purpose a wall of masonry is usually erected over the arches; the weight of which must press unequally on the several sections, according to the horizontal breadth and height of the columns, which are superincumbent on them: through the terminations of the wedges A, B, C, D, &c. (Fig. 7.) draw the lines A *a*, B *b*, C *c*, &c. perpendicular to the horizontal line P Q, and in the lines *a* A, *b* B, *c* C, *d* D, &c. produced, if necessary, take the line *b* B of such a magnitude, that it shall be to the line *a* A in the same proportion, which the weight of the wedge B, with the weight above it, bears to the weight of the wedge A, with the weight above it, and so on; each line c C, d D, &c. being taken proportional to the weight of the respective wedges, including the weights superincumbent on them. In the next place we are to ascertain from these conditions, together with the angle of the first section or A°, the other angles B°, C°, D°, &c. so that the entire are, when loaded with the weights, denoted by the lines A a, B b, C c, &c. may be equally balanced in all its parts. Admitting that the angle of any section D°, can be ascertained from having given the weight of the section D, and the pressure on it, together with the inclination of the abutment of the preceding section, or the angle of the abutment C to the vertical, it follows as a consequence, that, by the same method of inference, the angles of all the sections will be successively obtained from the angle of the first section, and the initial pressure, which are given quantities in the construction of every arch.

Suppose, therefore, any number of the sections A, B, C, to have been balanced by the requisite adjustments. It is required to determine the angle of the next section or D°, on the following conditions.

1st. That the direct pressure on the abutment F C of the section D, from the preceding sections shall be given, equal to the oblique pressure on the line I F, denoted by the line S B, which is drawn perpendicular to the line F C produced. (Fig. 8.)

2d. That the weight of the section D, including the weight superincumbent on it, shall also be given: let this weight be represented by the line D d.

3d. The angle c C H being the inclination of the line F C, or the abutment of the preceding section C to the vertical, is also a given quantity.

Referring to figure 3, we observe, that the adjustment of the angle D° depends on the equality of the line H D representing the force arising from gravity, and urging the section to descend along the line I D, and the force B F, which impels it in the opposite direction D I (Fig. 3.) In this case, F B is the sine of the angle * B S F = D° to a radius = B S, and H D is the cosine of the angle H D *d* to a radius = D *d*. Through the point D draw D A parallel to C F; so shall the angle A D *d* be equal to the angle H C *c*, which is given by the conditions: moreover, the angle I D A is the unknown angle of the section D°, which is required to be determined, and the inclination of the line I D to the vertical line D *d*, when constructed, will be equal to the sum of the angles A D *d* + I D A = I D *d*.

B S represents the pressure on the section D (Fig. 8.) From these conditions the angle required D° or I D A, and the inclination of the abutment I D *d* are determined by the following construction: through the point D, which terminates the base of the section D, draw the line D *d* perpendicular to the horizon and equal to the given line which represents the weight of the section D: with the centre D and distance D *d* describe a circle: through the point D draw the line D A parallel to C F, cutting off an arc A *d*, which measures the angle A D *d* equal to the given angle *c* C H. Through the points D and *d*, draw the indefinite lines D Z *d* Y perpendicular to the radius D *d*, and through the point A draw the indefinite line A W parallel to *d* Y. In the line A W set off A N equal to the given line B S, and through the points N and D draw the line D N, intersecting the circle in the point H:

* Because S B, S F are by construction perpendicular to the lines F Z, I F respectively; consequently the inclination of the lines F Z, I F, that is, the angle of the section D°, will be equal to the inclination of the lines S B, S F, or the angle B S F.

from the point A set off an arc AI, equal to the arc GH, and through the points I and D draw the line IDQ. The angle IQF or IDA will be the angle of the section D, which was required to be determined by the construction, and I D d will be the inclination of the abutment D I of the section D to the vertical line D d. The demonstration is as follows: through the point S draw SF perpendicular to I Q, and SB equal to the line denoting the given pressure on the line F I, and perpendicular to F Q; also through G draw GO perpendicular to DG, and through H draw HP perpendicular to DG; also produce DN till it intersects d Y in L: produce N A till it intersects D d in the point F, and through d draw d H perpendicular to DI. It is to be proved that the cosine of the angle IDd to the radius Dd, is equal to the sine of the angle I D A or G D O to the radius B S, or that the line D H or DK is equal to the line BF.

By the similarity of the triangles P H D, N D F, as P H : DP :: DF : NF; but by the construction NF = BS + AF. Wherefore PH : DP :: DF : BS + AF, and PH × BS + PH × AF = DF × DP, or PH × BS = DF × DP − PH × AF: dividing both sides by the radius Dd, $\frac{PH \times BS}{Dd} = \frac{DF \times DP - PH \times AF}{Dd}$.

But because the angle I D A is equal to the angle G D O, DF × DP is the rectangle under the cosines of the angles ADd, IDA, and P H × AF is the rectangle under the sines of the said angles: wherefore, by the principles of trigonometry, the difference of those rectangles divided by the radius Dd, that is, $\frac{DF \times DP - PH \times AF}{Dd}$ will be the cosine of the sum of the angles ADd + ADI, or the cosine of IDd = DH or DK. But it has been shewn that $\frac{PH \times BS}{Dd} = \frac{DF \times DP - PH \times AF}{Dd}$; therefore

$\frac{PH \times BS}{Dd} = DK$: and by the similar triangles PHD, BSF, as PH : DH or Dd :: BF : BS; consequently, $BF = \frac{BS \times PH}{Dd}$. But $\frac{BS \times PH}{Dd} = DK$; therefore BF is equal to DK, which was to be proved.

The angle of the section D° will be readily computed from the value of the Fig. 8. tangent $GO = \frac{Dd \times DF}{NF} = \frac{Dd \times DF}{BS + AF} = \frac{DF}{BS + AF}$ to radius $= 1$; or if the given angle $HCc = ADd$ be represented by V^c, we shall have $DF = Dd \times \cos. V^c$, and $AF = Dd \times \sin. V^c$. Wherefore, putting $BS = r$, the tangent in the Tables of the angle $D^\circ = \frac{Dd \times \cos. V^c}{r + Dd \times \sin. V^c}$.

To express the solution of this case generally by analytical values, let the weight of the first section or A be denoted by the letter w; and let the angle of the first section $= A^\circ$. The initial pressure $= p = \frac{w}{2 \times \sin. \frac{1}{2} A^\circ}$, and let the given weights of the successive sections, (Fig. 7.) including the weights superincumbent, be denoted by the letters a, b, c, d, &c. respectively, which are represented in the figure by the lines Aa, Bb, Cc, Dd, &c. the angles of each section, and the pressures on the section next following are as they are stated underneath, for adjusting the arch to equilibration.

Sections.	Weights of the Sections.	Tangents of the Angles of the Sections.	Angular Distances of the Abutments from the vertical Line.	Entire Pressures on the Sections next following.
A	a	tang. $A^\circ =$ tang. A°	$V^a = V^a = \frac{1}{2}A^\circ$	$p \times \cos. A^\circ + a \times \sin. V^a = p$
B	b	tang. $B^\circ = \frac{b \times \cos. V^a}{p + b \times \sin. V^a}$	$V^b = V^a + B^\circ$	$p \times \cos. B^\circ + b \times \sin. V^b = q$
C	c	tang. $C^\circ = \frac{c \times \cos. V^b}{q + c \times \sin. V^b}$	$V^c = V^b + C^\circ$	$q \times \cos. C^\circ + c \times \sin. V^c = r$
D	d	tang. $D^\circ = \frac{d \times \cos. V^c}{r + d \times \sin. V^c}$	$V^d = V^c + D^\circ$	$r \times \cos. D^\circ + d \times \sin. V^d = s$
E	e	tang. $E^\circ = \frac{e \times \cos. V^d}{s + e \times \cos. V^d}$	$V^e = V^d + E^\circ$	$s \times \cos. E^\circ + e \times \sin. V^e = t$

The application of these analytical values will be exemplified by referring to the case of an arch formed by sections, which are disposed according to the figure of any curve, when the columns which are built on the arches, terminate in an horizontal line; the given weights of the sections A, B, C, &c. (including the weights of the columns built upon them), denoted by the lines A a, B b, C c, &c. being as follows.

A a = the weight of the first section, (Fig. 7.) is represented by the number 2; B b = 2.76106, C c = 5.03844, and so on. The weight of the first section, which is denoted by the number 2, may be taken to signify 2 hundred weight, 2 ton, &c. all other weights being in proportion to it; the angle of the first or highest section is $5^\circ = A^\circ$, and $w = 2$. The initial pressure or $\frac{w}{2 \times \sin. 2^\circ 30'} = 22.9225 = p$: making $b = 2.76106$, we obtain from the preceding theorem, the tangent of $B^\circ = \frac{b \times \cos. V^a}{p + b \times \sin. V^a}$ = tang. $6^\circ 49' 31''$, wherefore the angle of the section B or $B^\circ = 6^\circ 49' 31''$; this added to $2^\circ 30'$ will give the inclination of the abutment of the section B to the vertical line $= 9^\circ 19' 31'' = V^b$; also since $p = 22.9255$, the pressure on the section $C^\circ = p \times \cos. B^\circ + b \times \sin. V^b = 23.21305 = q$, and thenceforward, according to the successive angles and the pressures on the sections next following, as they are entered in the Table No. II. entitled, A Table shewing the Angles of the Sections, &c. calculated from the given Weights, &c.

The lines a A, b B, c C, representing the given weights of the sections and the weights superincumbent on them, (if the line L a = 2 be subtracted from each) are nearly proportional to the versed sines of the arcs of a circle, increasing by a common difference of 5°; the curve of the arch will therefore be scarcely different from the arch of a circle. The third column of the second Table shews the several angles of the sections, which will create a sufficient force of counterpoise to the weight of the sections, together with the weights of the columns built upon them. It appears by inspecting this Table, that the angles of the sections first increase, reckoning from the highest or middle wedge, till the semiarc is augmented to about 55°; and afterwards decrease; plainly indicating that part of the arch which requires the greatest aid from the increased angles of the sections, as a counterpoise to the weight above them. If therefore the angles of the sections were constructed equal, as they usually are, the form of the arch being circular, and if a wall of solid masonry should be built upon it, terminating in an horizontal line or plane, it is clearly pointed out, what part of the arch would be the most likely to fail, for want of the requisite counterpoise of equilibrium; and although the fastenings should be sufficient to prevent the form of the arch from being immediately altered, the continuance of its constructed figure would depend on the resistance opposed by the fastenings to the stress arising from a defect of equilibrium, which acts incessantly to disunite the sections; a preponderance of this force, to a certain degree, would probably break the arch somewhere between 50° or 60° from the highest or middle section.

In adjusting the equilibrium of an arch, it is observable that the lengths of the bases which form the interior curve, usually termed the curve of the arch, are not among the conditions given, from which the weights or angles of the sections are inferred. A circumstance which renders the solution here given of the problem for adjusting equilibration, very general.

Whatever, therefore, be the figure of the interior curve, the bases of the sections which are disposed in this form, may be of any lengths, provided the weights and angles of the sections are in the proportions which the construction demands: observing only, that if the lengths of the bases should be greatly increased in respect to the depths, although in geometrical strictness, the properties of the wedge would equally subsist, yet when applied to wedges formed of material substance, they would lose the powers and properties of that figure: this shews the necessity of preserving some proportion between the lengths of the bases and depths of the wedges, to be determined by practical experience, rather than by geometrical deduction.

The following constructions and observations will further shew how little the equilibrium of an arch depends on the figure of the curve line by which it is terminated. All the properties of arches being (so far as the preceding constructions and demonstrations may be depended on) the consequences of the weights and pressure of the sections, acting without relation to the figure of any curve, so that arches may be constructed which terminate in a circular, elliptical, or any other curve, retaining the properties of equilibration indifferently in all these cases. To exemplify this principle by a simple case, let all the sections which form any arch be of equal weights, the angle of the first wedge, or A°, being $= 5^{\circ}$; and let it be required to ascertain the angles of the other sections, so that the pressures may be a counterpoise to their weights in every part. Assuming, therefore, as conditions given, (Fig. 9.) the angle of the first or highest wedge $A^{\circ} = 5^{\circ}$, and the weights of the several sections $= 1 = A a = B b = C c = D d$, &c. we have for the construction of this case the initial pressure $= \frac{1}{2 \times \sin. 2^{\circ}\, 30'} = p = MX$, the angle of the abutment A or V^{a} $= 2^{\circ}\, 30'$. From these data the angles* of the sections will be

* If the weighes of the sections A, B, C, D, &c. which are denoted by the lines *a*, *b*, *c*, *d*, &c. were made equal to 1.0077, 1.0312, 1.0719, &c. as they are stated in the Table

constructed according to the solution in page 27, from which the following results are derived: $\frac{\cos. V^a}{p + \sin. V^a} = \text{tang. } B^\circ = 4^\circ\, 57'\, 40''$, therefore $B^\circ = 4^\circ\, 57'\, 40''$, which being added to $2^\circ\, 30'$, the sum will be equal to the angle $HBb = V^b\ 7^\circ\, 27'\, 40''$. C° is found by the same theorem to be $= 4^\circ\, 51'\, 10''$, and $V^c = 12^\circ\, 18'\, 50''$, and so on, according to the statements in the Table No. 3.

OIVIO (Fig. 10.) is the arc of a circle drawn from the centre O, and bisected by the vertical line VO. PWQ (Fig. 11.) is the arch of a circle drawn from a centre any where in the line VO, produced if necessary; TYS (Fig. 12.) is the arch of an ellipse, the lesser axis of which coincides with the line VO. WXZ (Fig. 13.) is a catenarian or any other curve which is divided by the vertical line VO into two parts, similar and equal to each other. These three curves form the interior figures of the three arches, the exterior boundaries of which are of any figures which make the semiarches on each side of the vertical line VO similar and equal.

In the next place, the circular arc OIVIO is to be divided into arcs, by which the angles of the sections in the three interior arches are regulated. For this purpose, from the point V on either side thereof, set off an arc $VG = 2^\circ\, 30'$; set off also from G the arc $GK = 4^\circ\, 58'$, omitting the seconds, as an exactness not necessary: and the subsequent arcs KF, FI, &c. according to the dimensions in the schedule annexed, extracted from the 3d Table, to the nearest minute of a degree.

No. I. The points O, R, Q, P, would all coincide in the point O; if the weights of the sections should be assumed greater than they are stated in the Table No. I, the points RQP would be situated between the points O and V.

Sections.	Arcs.	Angles. ° ′
½ A	V G	2 30
B	G K	4 58
C	K F	4 51
D	F I	4 41
E	I M	4 21
F	M N	4 12
G	N O	3 55
H	O Q	3 39
I	Q R	3 22

The arcs (Fig. 10, et sequent.) V G, G K, K F, &c. having been thus set off, according to the angles of the sections A, B, C, &c. through the points G, K, F, I, in the circular line O I V I O, &c. draw the lines G O, K O, F O, I O, &c. dividing the three arches into sections or wedges A 1, B 1, C 1, &c. A 2, B 2, C 2, and A 3, B 3, C 3, &c. Which wedges, when formed of material substance, will become so many arches of equilibration, if the weights of each section be equal to the weight of the highest or middle wedge A. This construction is not intended to point out any practical mode by which the forms of wedges, that constitute arches of equilibration may be delineated, but merely to shew, by a very simple case, and at one view, how much the curves of arches may be varied, while the properties of equilibrium still remain the same, in a geometrical sense; although in the practical constructions of arches, the greater curvature of an arch allows greater latitude for the unavoidable errors in execution, and for those which are the consequences of the imperfect nature of the materials used in the construction. The figures in which the sections are here disposed have been adopted for the purpose of shewing in what manner the wedges in the several arches A 1, B 1, C 1, &c. A 2, B 2, C 2, &c. are adjusted to equilibrium by means of the division of the external arc, according to the angles inferred from the solution of this case, (page 30) in which the weights of the sections are all equal, producing the angles which are entered in the Table No. III. and in the Schedule, page 32.

Admitting, then, for the sake of establishing a case, that the specific gravity of the sections should be capable of adjustment, so that their weights may be equal, although their magnitudes be different; admitting also, that in increasing or diminishing the volumes of the sections, the angles are continued invariably the same; the lengths of the bases may be lessened or augmented in any proportion that is required, the equilibrium of the section remaining unaltered. Thus, if it should be proposed, that any number of the sections in the arch (Fig. 13.) A 3, B 3, C 3, shall occupy a length denoted by the curve (Fig. 14.) *i v i*, and that the joinings of the sections should intersect the curve in the points *i f k g g k f i*, as represented in the figure.

Through the point *g* draw the line *g* G parallel to G O, and through *k* draw *k* K (Fig. 14.) parallel to K O, and through *f* draw *f* F parallel to F O, and so on: the sections A, B, C, &c. in Fig. 14, admitting their weights to be equal, would form an arch of equilibration: the same consequences will follow if this construction is applied to the rectilinear or flat arch, (Fig. 15.) if it be allowed to use that term, meaning the figure terminated by two arched surfaces when their curvature is diminished to nothing, and coinciding with two plane surfaces parallel to the horizon; such is the rectilinear figure P Q, P Q, Fig. 15. Suppose this figure to be divided into wedges that have the properties of equilibration, by which the force of pressure impelling them upward is counterbalanced by their weight; and suppose the joinings of the wedges are required to pass through the points M, N, O, Q, &c. Through M draw T M parallel to G O, and through N* draw R N parallel to K O and through O draw W O parallel to F O′, &c. If wedges of these forms are disposed on each side of the vertical line V L, the extreme sections being supported by the abutments P Q, P Q; the whole will be sustained in equilibrio, on the condition that the weight of each section is equal to the weight of the section A.

But supposing the wedges to be formed, as is usually the case, of solid substances which are of the same uniform specific gravity, to make their weights equal, the areas of the figures must be adjusted to equality; which requires the solution of the following problem: Having given the angles of any of the wedges as above stated, and having given the area of the wedge A $= \frac{\overline{Tt + Mm} \times VL}{2}$, to ascertain the lengths of the upper surfaces R T, R W, P W, and of the bases M N, N O, O Q, of the wedges B, C, D, &c. So that the areas R N T M, W O R N, P Q W O, may be equal to the area T t M m, with the condition that the angles of each wedge shall remain unchanged; that is, the lines T M, R N, W O, &c. shall be parallel to the lines G O, K O, F O, I O, &c. respectively: to obtain the lengths of the lines R T, M N, which terminate the wedge B, according to these conditions, make the perpendicular distance V L $= r$, the area T M t m = A; cotang. M N R — cotang. L M T = D; then R T $= \frac{2A + r^2 \times D}{2r}$, and M N $= \frac{2A - r^2 \times D}{2r}$.

* Fig. 10.

As an illustration of this theorem, let the angle of the middle wedge A (Fig. 15.) be assumed = $30°$: if the weights of all the other sections are equal to that of A, the successive angles as determined by a preceding construction, in page 24, &c. will be as follows, $B° = 23° 47' 38''$: and in like manner, the angles C°, D°, and E° are found to be as they are stated in Table IV; the wedges being constructed according to these angles will give the following results: the angle $LMT = 90° + V^a = 105°$; $MNR = 90° + 38° 48' := 128° 48' = 90 + V^b$; $NOW = 90° + 53° 16' = 143° 16' = 90 + V^c$, and so on. From hence we obtain the lengths of the lines TR, MN. When the area RNTM is equal to the area $Tt\ Mm = A$. Let the perpendicular distance VL $= r = 2$ feet; and suppose the radius = 5 feet, then the angle TOt being $= 30°$, $Tt = 2.679492$, and $Mm = 1.607695$, and the area of the section $A = \overline{2.679492 + 1.607695} \times \frac{2}{2} = 4.287187 = A$. Cotang. $105°$ — cotang. $128° 48'$ by the Tables $= .5360714 = D$; wherefore the line $TR = \frac{2A + r^2 D}{2r} = 2.679492$, and $MN = \frac{2A - r^2 D}{2r} = 1.607522$.

The dimensions of the sections C, D, &c. are determined from the same rule, and are as underneath.

Sections.	Lengths of the upper Surfaces.	Lengths of the Bases.	Oblique Lines, or Secants.
A	$Tt = 2.679492$	$Mm = 1.607695$	$VL = 2.$
B	$TR = 2.679665$	$MN = 1.607522$	$TM = 2.0705524$
C	$RW = 2.679548$	$NO = 1.607639$	$RN = 2.5662808$
D	$WP = 2.679077$	$OP = 1.608110$	$WO = 3.3439700$
E	$PX = 2.680363$	$PY = 1.606824$	$PQ = 4.2508096$

According to the geometrical construction for adjusting the equilibrium of an arch by the angles between the sides of the sections or wedges, the architect will be enabled to distribute the mass of materials, whether they consist of stone or iron, of which the arch is intended to consist, among the sections, in any proportion that may best contribute to strengthen and embellish the entire fabric, establishing the equilibrium of the arch at the same time. This principle of construction would be of use, more particularly where circumstances may require that the equilibrium should be adjusted with great exactness. Supposing that according to the plan of the structure, the angles of the sections are equal to each other; if the mass or weight which the adjustment of the equilibrium allots to the sections near the abutments, should be diffused over too great a base; or may be, for other reasons, independent of any consideration of equilibration, judged too weak to support the superincumbent weight with security, this inconvenience would be remedied by adding such a quantity of materials to the weaker sections, as may enable them to support the weights or loads they are required to bear, and afterwards adjusting the angles of these sections, so as to form an arch of equilibration, according to the rules which have been given, (page 27). Or perhaps it might be expedient to arrange, in the first instance, the quantity of materials which ought to be allotted to the several sections of the entire arch, and afterwards to adjust the angle of each section, so as to form the equilibrium: suppose, for instance, the form of the arch be such as is represented in the figure 16: V A B C D, &c. is a circular arc drawn from the centre O and with the radius O V. Let the bases of the sections be terminated by the arcs A B, B C, C D, each of which subtends, at the centre O, an angle of 1°. Through the points

A, B, C, D, &c. draw the indefinite lines A a, B b, C c, perpendicular to the horizon, and suppose the masses or weights allotted to the sections would give a sufficient degree of strength to the entire structure, when the weight of the section contiguous to the abutment is three times the weight of the first section, and the intermediate weights are increased by equal differences, from 1 to 3. If, therefore, the number of sections should be fixed at 49, so as to make the angle of the arch when viewed from the centre $O = 49°$, the weight of the highest or middle section being assumed equal to unity, the weight of the section B or B b = 1.083333, Cc = 1.166666, &c.; and finally the weight of the section next the abutment, or Z = 3; as they are stated in the Table No. V. in the column entitled, Weights of the Sections. Since, therefore, the angle A° is by the supposition = 1°; the initial pressure, or $\frac{1}{2 \times \sin. \frac{1}{2} A°} = 57.29649 = p$. And because $V^a = 30'$, and b is the line denoting the weight of the section B = 1.083333, according to the rule for determining the angles of the sections, so as to form an arch of equilibration, $\frac{b \times \cos. V^a}{p + b \times \sin. V^a}$ = the tangent of 1° 34′ 58″ = B°; which angle being added to 30′ or V^a, the sum will be = 1° 34′ 50″ = V^b, or the inclination of the abutment to the vertical, of the section B; from whence we obtain the entire pressure on the next section $= p \times \cos. B° + b \times \sin. V^b = 57.31593 = q$, and $\frac{c \times \cos. V^b}{q + c \times \sin. V^b}$ = tang. 1° 9′ 54″; therefore the angle of the third section C° = 1° 9′ 54″, and so on. The angles of the sections D°, C°, and the corresponding angles of the abutments are entered in Table V. From the angles of the abutments determined by these calculations, the practical delineation of the sections will be extremely easy: having described the arc of a circle with any radius O V, and the several arcs A B, B C, C D, &c. being set off equal to 1°, the positions of the successive abutments which terminate the sections on each side will be found by taking the difference between the inclination of the abutment to the vertical, and the angle subtended by the semi-arc at the centre of the circle, if this difference be put = D, that is, to exemplify for the abutment F Y, if the difference of the angles V P F — V O F, or P F O be made = D; then the line $OP = \frac{\text{sin. } D \times DF}{\text{sin. } VPF}$: consequently the length of the line O P being ascertained, through P and F draw the line P F Y, which will be the position of the abutment on which the section F rests. And a similar construction will determine the positions of all the lines E X, D W, C Q, &c.; when the sections form an arch of equilibration according to the conditions given. By this rule the lines O T, O S, O R, &c. (Fig. 16.) are found, according to the following Table, O V = O F being put = radius = 1000.

Sections.	Distances from the Centre O.	Sections.	Distances from the Centre O.
A	OO = 0	O	= 338.00
B	OT = 52.651	P	= 351.10
C	OS = 90.009	Q	= 363.17
D	OR = 123.22	R	= 374.12
E	OQ = 153.38	S	= 383.21
F	OP = 181.90	T	= 393.25
G	= 207.48	U	= 401.33
H	= 230.95	V	= 408.62
I	= 252.44	W	= 415.13
K	= 273.36	X	= 420.78
L	= 290.93	Y	= 426.15
M	= 307.95	Z	= 429.96
N	= 323.53		

The Tables No. I. II. III. IV. V. subjoined to these pages, have been calculated rigidly from the rules in pages 14 and 27; a column containing the weights of the semiarcs, or the weights of $\frac{1}{2}$ A + B + C + D, &c. has been added to each of the Tables, for the purpose of comparing them with the general rules for approximating to the correct values inserted in page 19. The calculations in the Tables are expressed to five or six places of figures, and the results of the approximate rule 5, which is $S = Z \times \sin. V^2$, coincides with the correct values in the Tables to four or five places, including the integers; the calculations made from the other rules, which include the horizontal force, approximate to the true values the more nearly, as the angles of the sections are smaller; but in any, except very extreme cases, they are sufficiently correct for all practical purposes; which will make the use of these approximations preferable to the troublesome calculations which are required for inferring the correct values from the original rules.

That a judgment may be formed of the errors to which these approximate values are liable, the Table No. VI. is added, containing the comparative results therein stated.

The examples in the Table No. VI. have been taken from the Table No. II. in which the angles are calculated to the nearest second of a degree; and the numbers to be 6 or 7 places of figures: an exactness not necessary, except for the purpose of comparing the results arising from the different rules for computing.

Table No. I.

Shewing the weights of the several sections or wedges which form an arch of equilibration, when the angle of each section is 5°; and the weight of the highest or middle wedge is assumed = 1. Also shewing the pressures on the lowest surface of each section, considered as an abutment.

The initial pressure $= \frac{w}{2 \times \sin. 2^\circ 30'} = 11.4628 = p$

The lateral or horizontal pressure $= \frac{w}{2 \times \text{tang. } 2^\circ 30'} = 11.4519 = p'$

Sections.	Angles of the Sections.	Angles of Inclination to the vertical Line of the Surface, on which each Section rests.		Weight of each Section.	Pressure on each successive Wedge considered as an Abutment.	Weights of the Semi-arcs, being the successive Sums of the Weights in the 4th Column, deducting from each Sum the Weight of $\frac{1}{2}$ A = .5.
	°	°	′			
A	5	2	30	1.	11.4628	.5
B	5	7	30	1.0077	11.5507	1.5077
C	5	12	30	1.0312	11.7300	2.5389
D	5	17	30	1.0719	12.0076	3.6108
E	5	22	30	1.1328	12.3954	4.7436
F	5	27	30	1.2180	12.9098	5.9616
G	5	32	30	1.3341	13.5775	7.2957
H	5	37	30	1.4916	14.4346	8.7873
I	5	42	30	1.7064	15.5325	10.4937
K	5	47	30	2.0038	16.9508	12.4975
L	5	52	30	2.4268	18.8116	14.9243
M	5	57	30	3.0515	21.3136	17.9758
N	5	62	30	4.0530	24.8284	22.0288
O	5	67	30	5.6547	29.9582	27.6835
P	5	72	30	8.6830	38.1254	36.3665
Q	5	77	30	15.3661	52.9822	51.7326
R	5	82	30	35.4777	87.9547	87.2103
S	5	87	30	175.7425	263.1952	262.9528

Table No. II.

Shewing the angles of the sections which form an arch of equilibration, calculated from the given weights of the sections, including the weights of the columns built upon them, terminating in a right line parallel to the horizon; the weight of the first section, or $A = 2 = w$.

The initial pressure $= p = \frac{w}{2 \times \sin. \frac{1}{2} A^{\circ}} = 22.92558$

The horizontal force or pressure* $= p' = \frac{w}{2 \times \text{tang.} \frac{1}{2} A^{\circ}} = 22.903766$

Sections.	Given Weights of the Sections.	Angles of the Sections.			Angles of Inclination to the Vertical of the lower Surfaces of each Section, considered as Abutments.			Total Pressure on the Abutment, or on the Section next following.	Weights of the Semi-arches, being the successive Sums of the Weights in the 2d Column, deducting from each Sum the Weight of ½ A.
		°	′	″	°	′	″		
A	2.00000	5	0	0	2	30	0	22.92558	1.00000
B	2.76106	6	49	31	9	19	31	23.21305	3.76106
C	5.03844	11	41	23	21	0	54	24.53843	8.79950
D	8.81484	16	32	41	37	33	36	28.89592	17.61434
E	14.06148	16	34	7	54	7	43	39.09063	31.67582
F	20.73844	12	15	55	66	23	38	57.19849	52.41426
G	28.79492	7	51	16	74	14	54	84.37357	81.20918
H	38.16960	4	53	24	79	8	18	121.55373	119.37878
I	48.79112	3	6	19	82	14	37	169.71989	168.16990
K	60.57864	2	2	16	84	16	54	229.88977	228.74854

* The invariable horizontal force, or pressure, is called, in the technical phrase, *the drift or shoot of an arch.*

Table No. III.

Containing the angles of the several sections, with the angles between the vertical line and the abutments, calculated from their weights, when they are assumed equal to the weight of the first section, the angle of which is given $= 5^\circ$; the weight of the first section, or $A = 1$.

$$\text{The initial pressure} = \frac{w}{2 \times \sin. \frac{1}{2} A^\circ} = 11.46279$$

$$\text{The horizontal force} = \frac{w}{2 \times \text{tang.} \frac{1}{2} A^\circ} = 11.45188$$

Sections.	Angles of the Sections.	Angles of Inclination to the Vertical of the lower Surfaces of each Section, considered as Abutments.	Total Pressure on the Abutment, or on the Section next following.	Weights of the Semiarches.
	° ′ ″	° ′ ″		
A	5 0 0	2 30 0	11.46279	.50000
B	4 57 40	7 27 40	11.54977	1.50000
C	4 51 10	12 18 50	11.72169	2.50000
D	4 40 40	16 59 40	11.97492	3.50000
E	4 27 20	21 27 0	12.30440	4.50000
F	4 12 10	25 39 10	12.70422	5.50000
G	3 55 30	29 34 40	13.16809	6.50000
H	3 38 40	33 13 20	13.68938	7.50000
I	3 21 40	36 35 0	14.26180	8.50000
K	3 5 40	39 40 40	14.87946	9.50000
L	2 50 20	42 31 0	15.53697	10.50000
M	2 36 20	45 7 20	16.22973	11.50000
N	2 23 20	47 30 40	16.95286	12.50000
O	2 11 20	49 42 0	17.70315	13.50000
P	2 0 20	51 42 30	18.47722	14.50000
Q	1 50 40	53 33 10	19.29421	15.50000
R	1 41 40	55 14 50	20.10753	16.50000
S	1 33 30	56 48 20	20.93711	17.50000
T	1 26 30	58 14 50	21.78074	18.50000
U	1 20 0	59 34 50	22.63742	19.50000
V	1 14 0	60 48 50	23.50502	20.50000
W	1 8 50	61 57 40	24.38295	21.50000
X	1 4 0	63 1 40	25.27003	22.50000
Y	0 59 40	64 1 20	26.16520	23.50000
Z	0 55 40	64 57 0	27.06773	24.50000
A	0 52 0	65 49 0	27.97699	25.50000
B	0 48 40	66 37 40	28.91436	26.50000
C	0 45 40	67 23 20	29.83490	27.50000
D	0 43 0	68 6 20	30.76038	28.50000
E	0 40 30	68 46 50	31.69023	29.50000
F	0 38 10	69 25 0	32.62449	30.50000

TABLE No. IV.

Shewing the angles of the sections, and the inclination of the abutments to the vertical line, calculated from the weights of the sections, when the angle of the first section is assumed $= 30°$, and the weight of each section is equal to the weight of the first section $= w = 1$.

$$\text{The initial pressure } p = \frac{w}{2 \times \sin.\ \frac{1}{2} A^{\circ}} = 1.931852$$

$$\text{The horizontal force } p' = \frac{w}{2 \times \text{tang.}\ \frac{1}{2} A^{\circ}} = 1.866025$$

Sections.	Angles of the Sections.	Angles of Inclination to the Vertical of the lower Surfaces of each Section, considered as Abutments.	Total Pressure on the Abutments, or on the Section next following.	Weights of the Semi-arches, deducting .5.
	° ′ ″	° ′ ″		
A	30 0 0	15 0 0	1.931850	.500000
B	23 47 38	38 47 38	2.394167	1.500000
C	14 28 4	53 15 43	3.119616	2.500000
D	8 40 25	61 56 8	3.966357	3.500000
E	5 32 31	67 28 39	4.871542	4.500000
F	3 46 33	71 15 32	5.807912	5.500000
G	2 43 23	73 58 56	6.762509	6.500000
H	2 2 46	76 1 41	7.7286	7.5000
I	1 35 23	77 37 4	8.7024	8.5000
K	1 16 8	78 53 12	9.6815	9.5000
L	1 2 8	79 55 21	10.6645	10.5000
M	0 51 38	80 46 59	11.9504	11.5000

Table No. V.

Shewing the angles of 49 sections, forming an arch of equilibrium, calculated from given weights of the sections, when the angle of the first section is 1 degree = A°; and the weight thereof is denoted by unity, the weights of the successive sections increasing by equal differences from 1 to 3, which is the weight of the 25th section of the semiarch Z.

The initial pressure $p = \frac{1}{2 \times \sin. \frac{1}{2} A^\circ} = 57.29649$

The horizontal force $p' = \frac{1}{2 \times \text{tang.} \frac{1}{2} A^\circ} = 57.29432$

Sections.	Weights of the Sections.	Angles of the Sections.	Angles of the Abutments.	Pressure on the Section next following.	Weights of the Semiarches.
		° ′ ″	° ′ ″		
A	1.000000	1 0 0	0 30 0	57.29649	.500000
B	1.083333	1 4 58	1 34 58	57.31593	1.583333
C	1.166666	1 9 54	2 44 51	57.36002	2.750000
D	1.250000	1 14 44	3 59 36	57.43352	4.000000
E	1.333333	1 19 28	5 19 4	57.54176	5.333333
F	1.416666	1 24 3	6 43 8	57.69030	6.750000
G	1.500000	1 28 28	8 11 37	57.88512	8.250000
H	1.583333	1 32 41	9 44 18	58.13190	9.833333
I	1.666666	1 36 35	11 20 53	58.43571	11.500000
K	1.750000	1 40 19	13 1 13	58.80511	13.250000
L	1.833333	1 43 39	14 44 52	59.24522	15.083333
M	1.916666	1 46 38	16 31 30	59.76187	17.000000
N	2.000000	1 49 13	18 20 44	60.36220	19.000000
O	2.083333	1 51 22	20 12 6	61.04895	21.083333
P	2.166666	1 53 4	22 5 10	61.83064	23.250000
Q	2.250000	1 54 18	23 59 29	62.71133	25.500000
R	2.333333	1 55 4	25 54 33	63.69574	27.833333
S	2.416666	1 55 21	27 49 55	64.78815	30.250000
T	2.500000	1 55 11	29 45 7	65.99240	32.750000
U	2.583333	1 54 33	31 39 41	67.31184	35.333333
V	2.666666	1 53 31	33 33 12	68.74902	38.000000
W	2.750000	1 52 4	35 25 17	70.30633	40.750000
X	2.833333	1 50 16	37 15 33	71.98551	43.583333
Y	2.916666	1 48 10	39 3 44	73.78788	46.500000
Z	3.000000	1 45 46	40 49 31	75.71422	49.500000

Table No. VI.

For comparing the pressures on the abutments, the weights of the semiarches, and the invariable horizontal or lateral force, calculated from the direct rules in the pages 13 and 27; with the pressures on the abutments, weights on the semiarches, and horizontal forces respectively, deduced by approximation, according to the general rules inserted in page 19.

In this Table, the weight of the highest or middle section is equal to the number 2, and the angle of the said section is $5°$; the weights of the several sections as they are stated in the Table II. By the approximate Rule I, the initial pressure $p = \frac{2}{2 \times \text{sin.} \frac{1}{2} A°} = 22.92558$. By the approximate Rule II, the horizontal force $p' = \frac{2}{2 \times \text{tang.}\ A°} = 22.90377$. S is the weight of the semiarch, and Z is the pressure on the abutment, the inclination of which to the vertical is $= V^z$.

Sections.	Angles of the Abutments, or V^z.	Rule III. Pressures on the Abutments.			Rule IV. Weights of the Semiarches.		
		Values of the Pressures Z, entered in the Table No. II.	Values of Z by the approximate Rule, $Z = p' \times$ Sec. V^z.	Differences.	Given Weights of the Semiarches, entered in Table II.	Weights by Approximation $S = p' \times$ Tang. V^z.	Differences.
	° ′ ″						
A	2 30 0	22.92558	22.92550	.00008	1.00000	1.	.0
B	9 19 31	23.21305	23.21052	.00253	3.76106	3.76107	.00001
C	21 0 55	24.53843	24.53560	.00283	8.79950	8 79896	.00054
D	37 33 36	28.89592	28.89120	.00472	17.61434	17.61204	.00230
E	54 7 43	39.09063	39.08690	.00373	31.67582	31.67351	.00231
F	66 23 38	57.19849	57.19580	.00269	52.41426	52.40850	.00576
G	74 14 54	84.37537	84.37180	.00357	81.20918	81.20260	.00658
H	79 8 18	121.55373	121.54790	.00583	119.37878	119.36970	.00908
I	82 14 37	169.71989	169.71090	.00899	168.16990	168.15830	.01160
K	84 16 54	229.88977	229.88290	.00687	228.74854	228.73900	.00954

TABLE No. VI. continued.

Sections.	Rule V. Weights of the Semiarches.			Rule VI. Invariable horizontal Force.		
	Given Weights of the Semiarches, entered in Table II.	Weights by Approximation $S = z \times$ Sin. Vz.	Differences.	Invariable horizontal Force $= \frac{2}{2 \times \text{Tang. } 2^{\circ}\ 30}$	Horizontal Force by Approximation $p = z \times$ Cos. Vz.	Differences.
A	1.00000	1.	.00000	22.90377	22.90311	.00066
B	3.76106	3.76147	.00041	22.90377	22.90626	.00249
C	8.79950	8.79990	.00040	22.90377	22.90625	.00248
D	17.61434	17.61473	.00039	22.90377	22.90723	.00346
E	31.67582	31.67647	.00065	22.90377	22.90592	.00215
F	52.41426	52.41092	.00334	22.90377	22.90483	.00106
G	81.20918	81.20665	.00253	22.90377	22.90514	.00137
H	119.37878	119.37512	.00366	22.90377	22.90499	.00122
I	168.16990	168.16712	.00278	22.90377	22.90514	.00137
K	228.74854	228.74630	.00224	22.90377	22.90444	.00067

N. B. The rules for approximating to the weights of the semiarcs, pressures on the abutments, and the invariable horizontal force, when applied to the Tables I. III. IV. and V. will be found to give results for the most part as exact as in the above calculations, which are formed from the Table No. II. The conditions on which this Table is founded being rather more complicated than in the other Tables, it was considered, on this account, to be the most proper test for examining the correctness of the approximate rules.

APPENDIX;

CONTAINING

NOTES AND CORRECTIONS.

Page 2, *Line* 1.

AFTER—" angular distance from the vertex,"—*add* measured by the inclination of the lowest surface to the vertical line.

Page 11, *the three last Lines.*

The weights are supposed to have been adjusted by geometrical proportions, but not mechanically determined with exactness.

Page 12, *Line* 25, *and in several other Places.*

In all the numerical computations of sines, cosines, and other lines drawn in a circle, the radius thereof is assumed equal to unity.

Page 17, *Line* 4.

The semiarch is understood to be that part of any arch which is comprehended between the vertical line and an abutment on either side.

Note, Page 18, *Line* 15.

Fig. 5. Let BOA $= A°$ represent the angle of the highest or middle section, so that the angle VOA shall $= \frac{1}{2} A°$: through any point I in the line OA draw the line KI perpendicular to OA, and supposing the weight of the wedge to be $= w$, let IK $= \frac{w}{2 \times \sin. \frac{1}{2} A°}$, or the initial pressure: resolve IK into two forces, namely, IO parallel, and DK perpendicular, to the horizon. By the similar triangles VOA, IDK, as

I K : I D :: radius to the cosine of K I D or V O A: it will follow that $\text{I D} = \frac{w \times \cos.\, \frac{1}{2} A^\circ}{2 \times \sin.\, \frac{1}{2} A^\circ} = \frac{w}{2 \times \text{tang.}\, \frac{1}{2} A^\circ}$, which is the measure of the invariable force, the direction of which is parallel to the horizon.

Page 26, *Line* 6, *at the word* "Through."

As the point S has not been yet determined by geometrical construction, *instead of*—"through the point S," &c.—*insert* through any point B in the line I Q draw the line B S perpendicular to the line C Q, and equal in length to the given line B S, which represents the pressure on the section D: and through the point S draw the line S F perpendicular to the line F I, and through G, &c.

Page 29, *Line* 6.

After—"till the semiarc is augmented to about 55°"—*add* (estimated by the inclination of the abutment to the vertical line).

ERRATA.

Page 7, line 18, *for* the line X P, *read* in the line X P.
—— 9, —— 20, *for* Fig. 1, *read* Fig. 1 and 2.
—— 11, —— 24, *for* then, *read* thus.
—— 11, —— 28, *for* direction, *read* directions.
—— 12, —— 11, *for* bissected, *read* bisected.
—— 23, —— 22, *for* over, *read* upon.
—— 24, —— 6 et alibi, *for* arc, *read* arch.
—— 29, —— 2 and 9, *for* weight, *read* weights.
—— 30, —— 12, *after* terminated *insert* ;
—— 30, in the note, *for* weighes, *read* weights.
—— 31, line 4, *read* $V^b = 7° 27' 40''$.
—— 32, —— 5, *for* 4° 21', *read* 4° 27'.
—— 35, —— 17, *for* T R = 2.679492, *read* 2.679665.
—— 37, —— 19, *for* 1° 34' 58", *read* 1° 4' 58".
—— 37, —— 20, *for* 1° 34' 50", *read* 1° 34' 58".
—— 38, —— 10, *for* D F, *read* O F.

The angles entered in the Tables I. II. III. IV. and V. are expressed to seconds of a degree, in some cases to the nearest ten seconds of a degree. These results will probably be found on examination, in most cases, correct to the degree of exactness here stated. Some errors may be expected to occur in the course of the long and troublesome computations which are required for forming these Tables. On a revisal, the undermentioned errata have been discovered, which the reader is requested to correct, together with any other which his own observation may have pointed out.

Section	Page 39.
C	*for* 90.009, *read* 90.127.
D	*for* 123 22, *read* 124.26.
E	*for* 153.38, *read* 154.03.
K	*for* 273.36, *read* 272.48.
S	*for* 383.21, *read* 384.10.
	Table II. Page 42.
D	*for* 37 33 36, *read* 37 33 35.

Section	Table III. Page 43.
D	*for* 16° 59′ 40″, *read* 16° 59′ 30″.
P	*for* 51 42 30, *read* 51 42 20.
	Table IV. Page 44.
C	*for* 53° 15′ 43″, *read* 53° 15′ 42″.
M	*for* 11.9504, *read* 11.6504.
	Table V. Page 45.
C	*for* 2° 44′ 51″, *read* 2° 44′ 52″.
D	*for* 3 59 36, *read* 3 59 35.

The angles opposite the sections F, G, K, &c. are affected by similar errors of 1″, which will appear by adding the angle of any section to the angle of the abutment preceding. The sum ought to be the angle of the abutment of the section.

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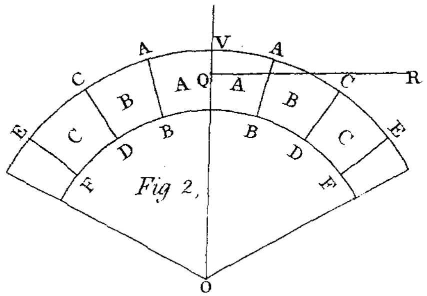
Plate I.



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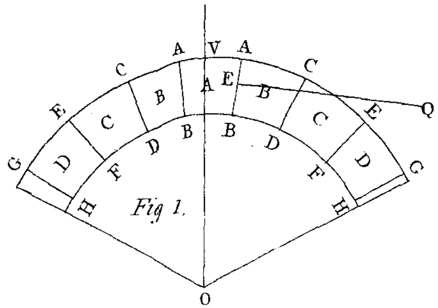


Fig 1.

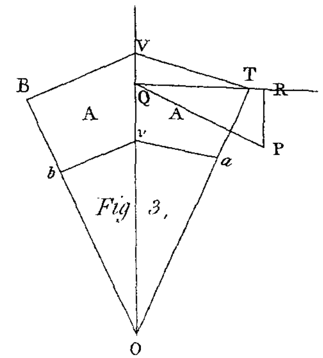


Fig 3,

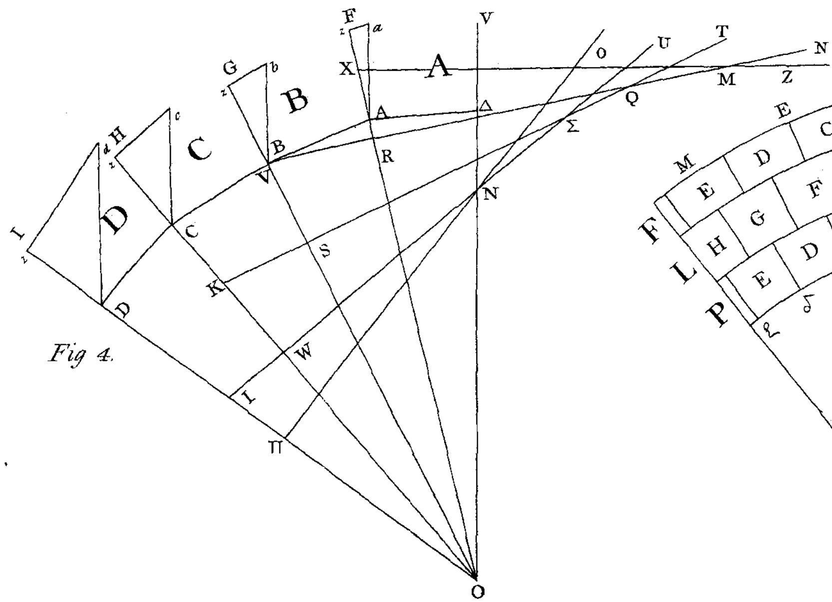


Fig 4.

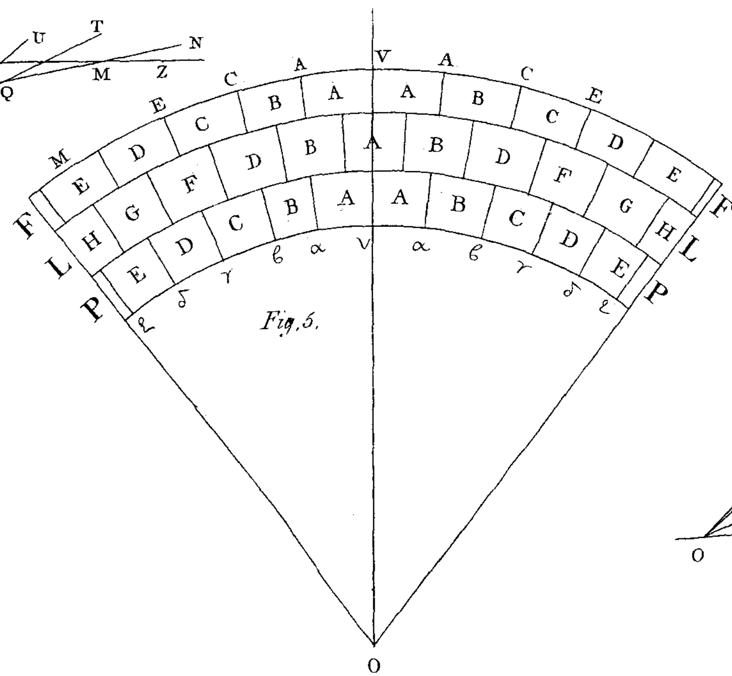


Fig.5.

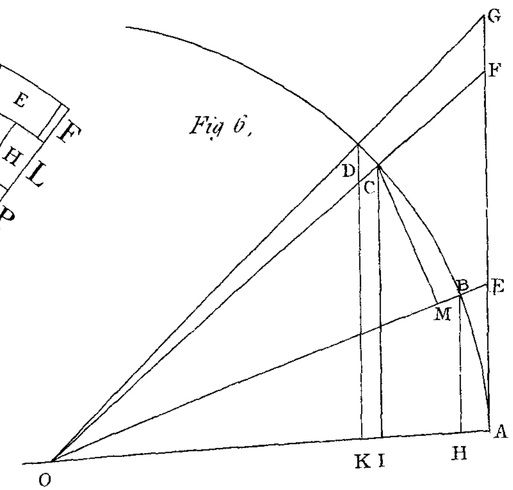


Fig 6,

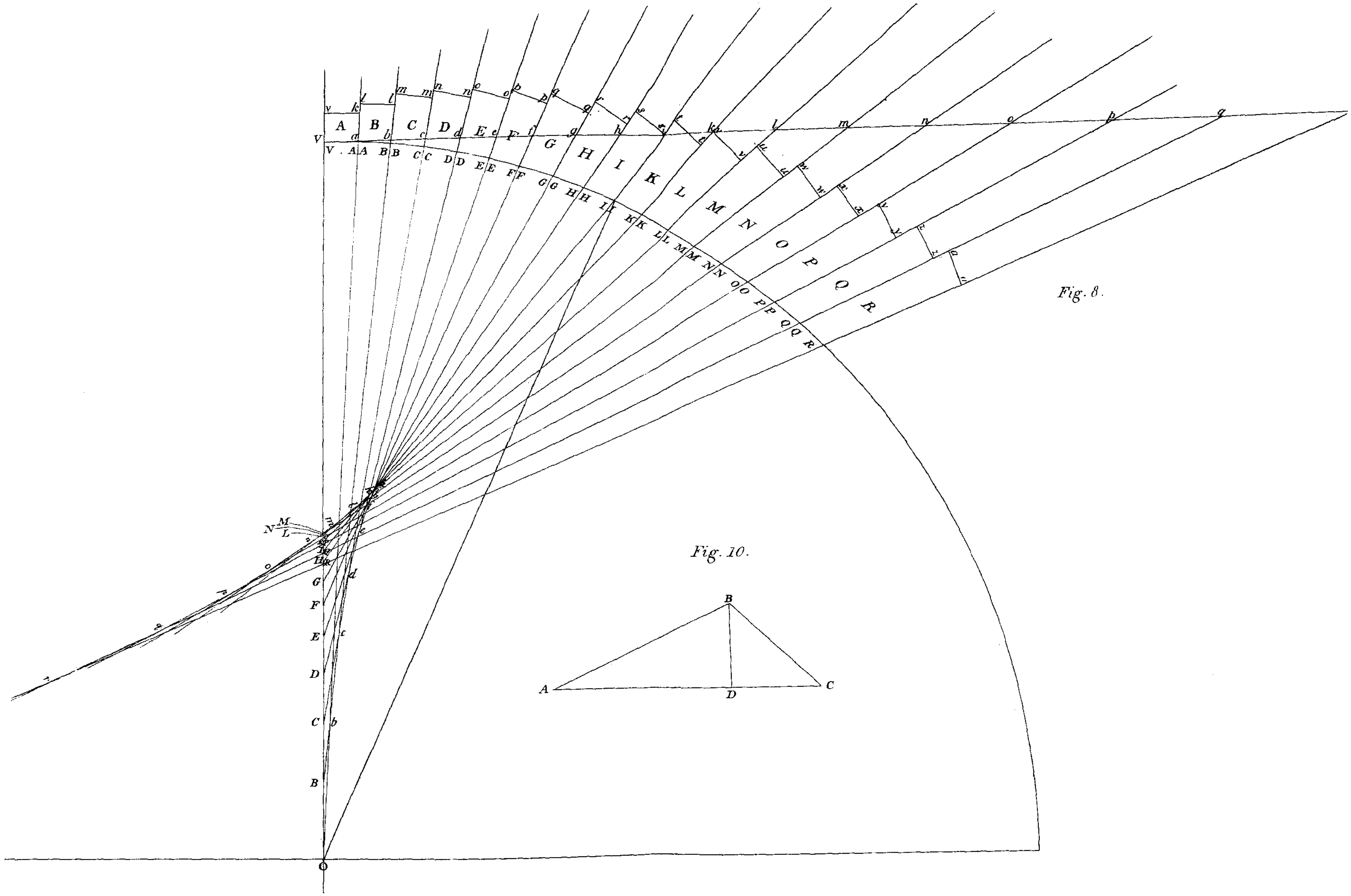
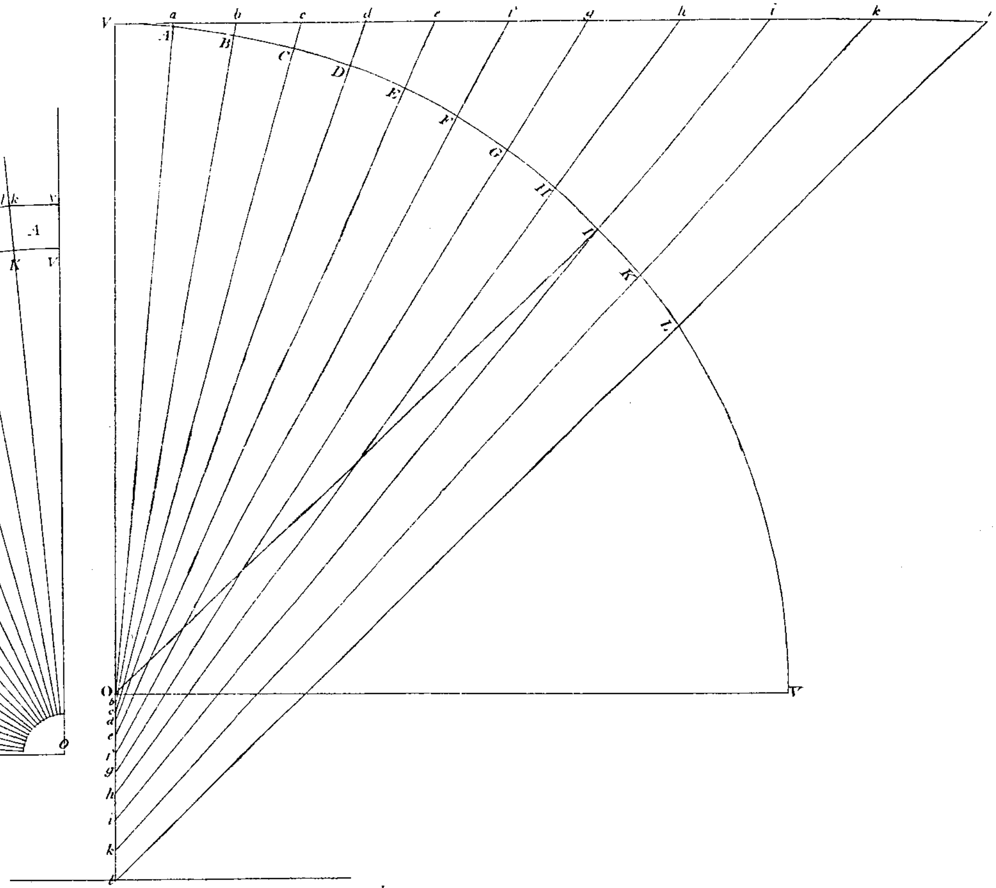


Fig. 8.

Fig. 10.

Fig. 9.



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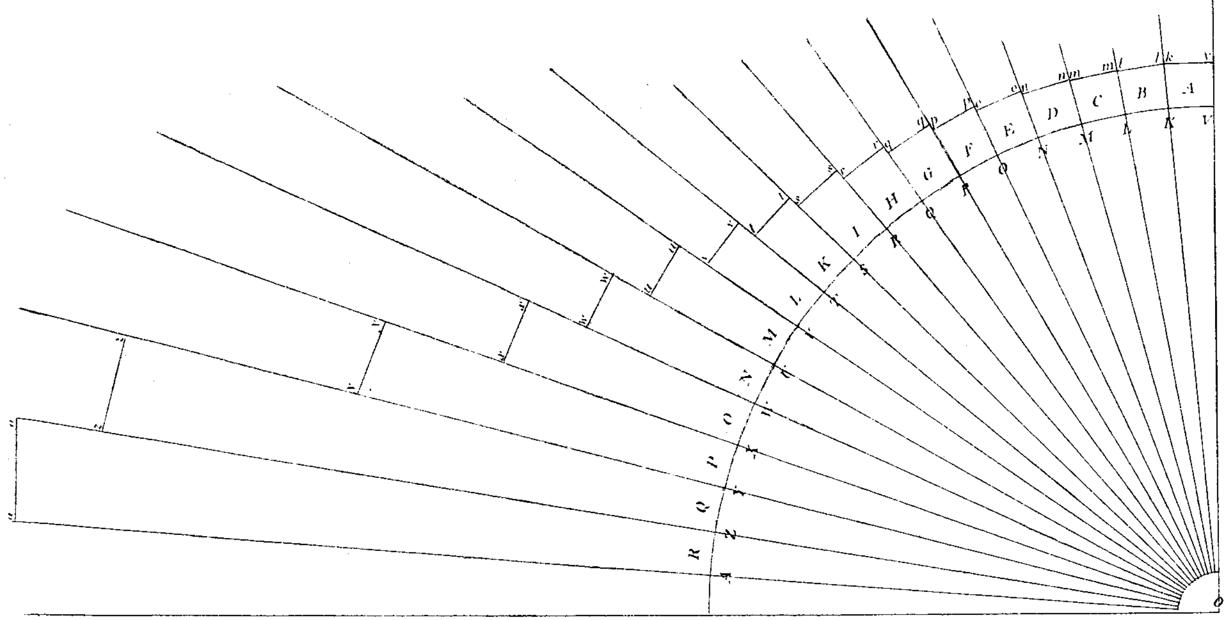
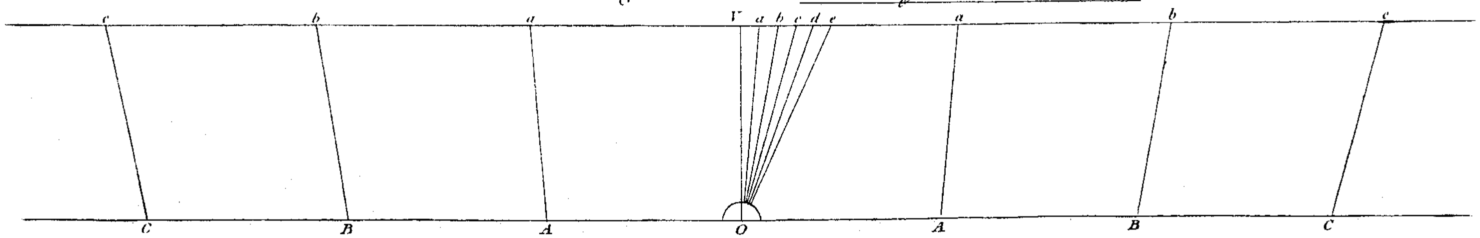


Fig. 11.



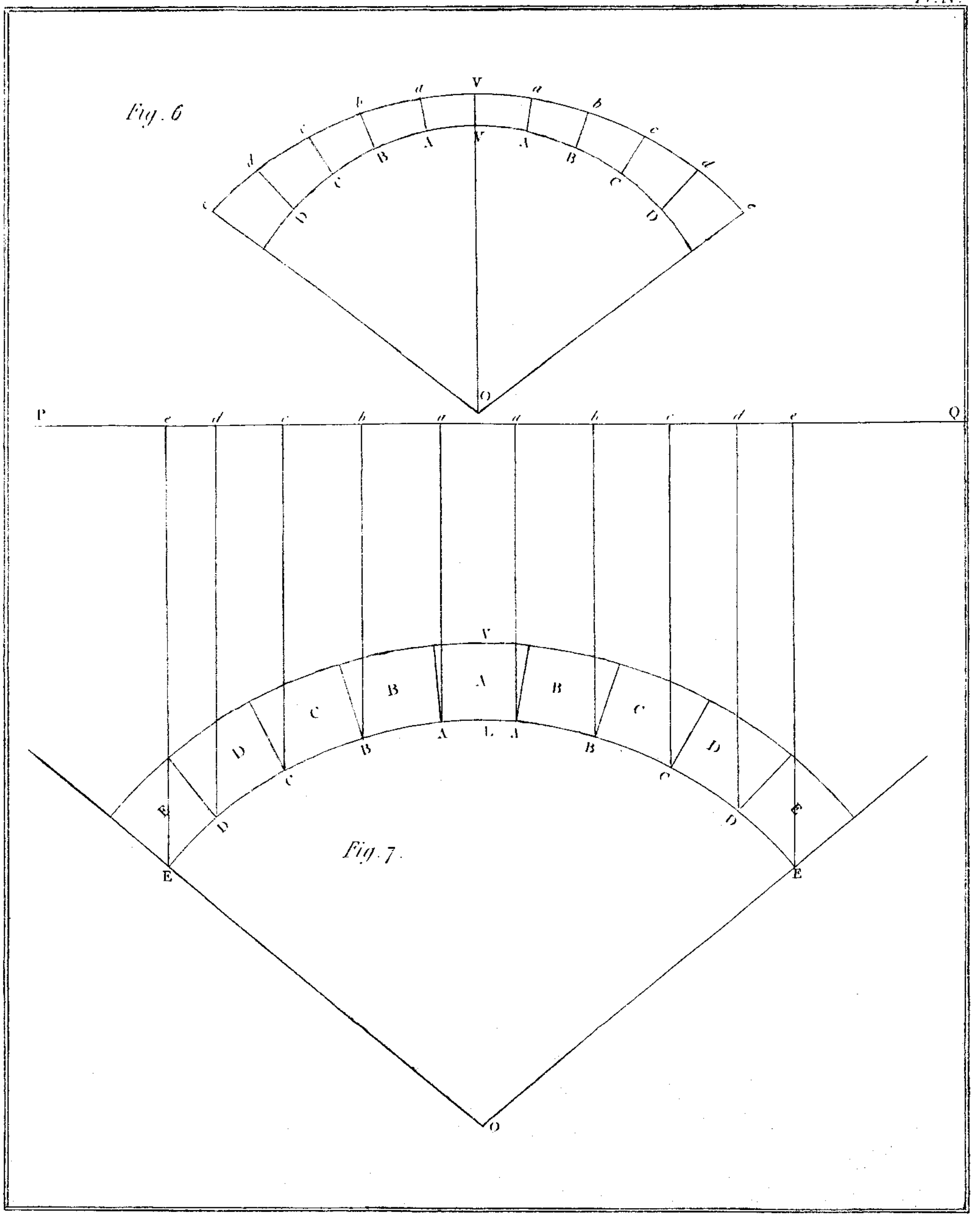


Fig. 6

Fig. 7.

Fig. 8.

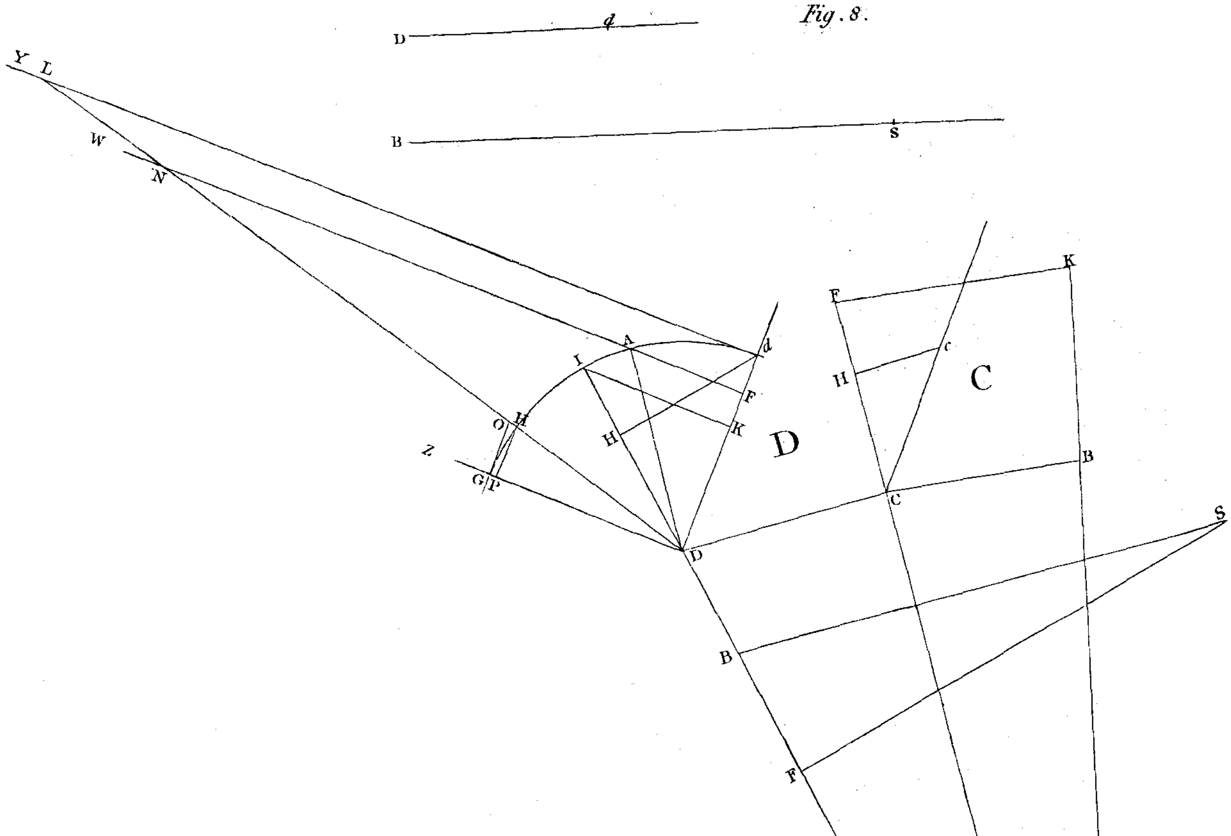
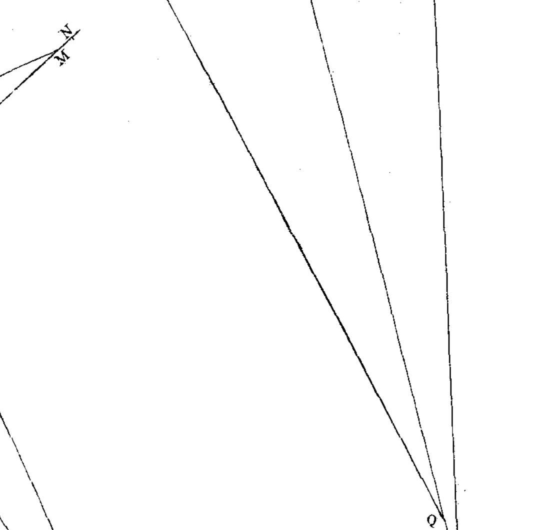
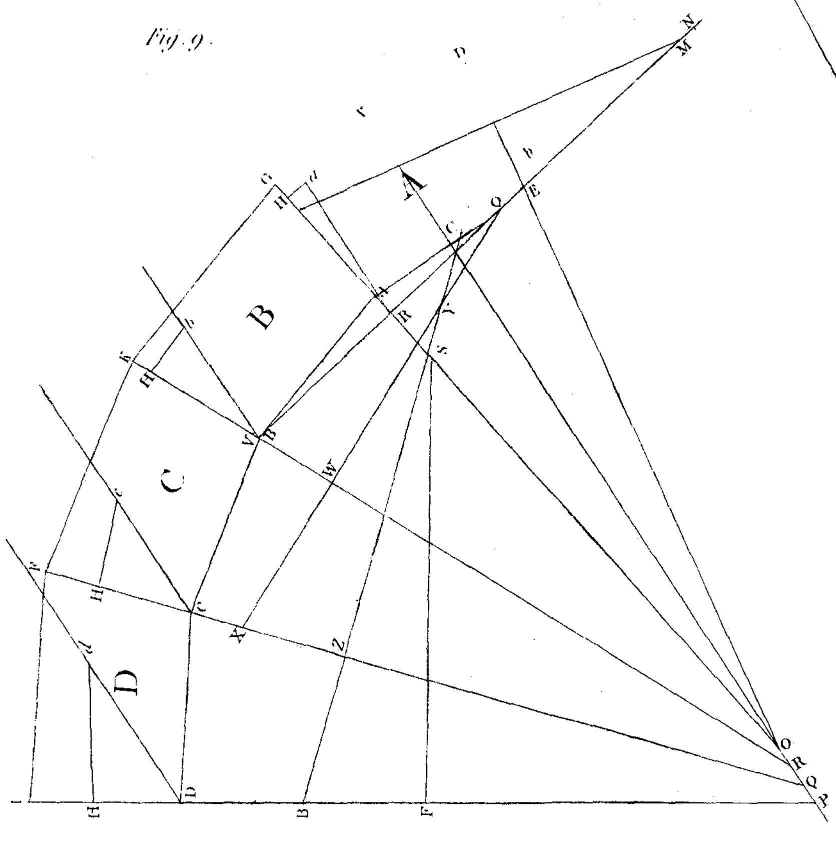
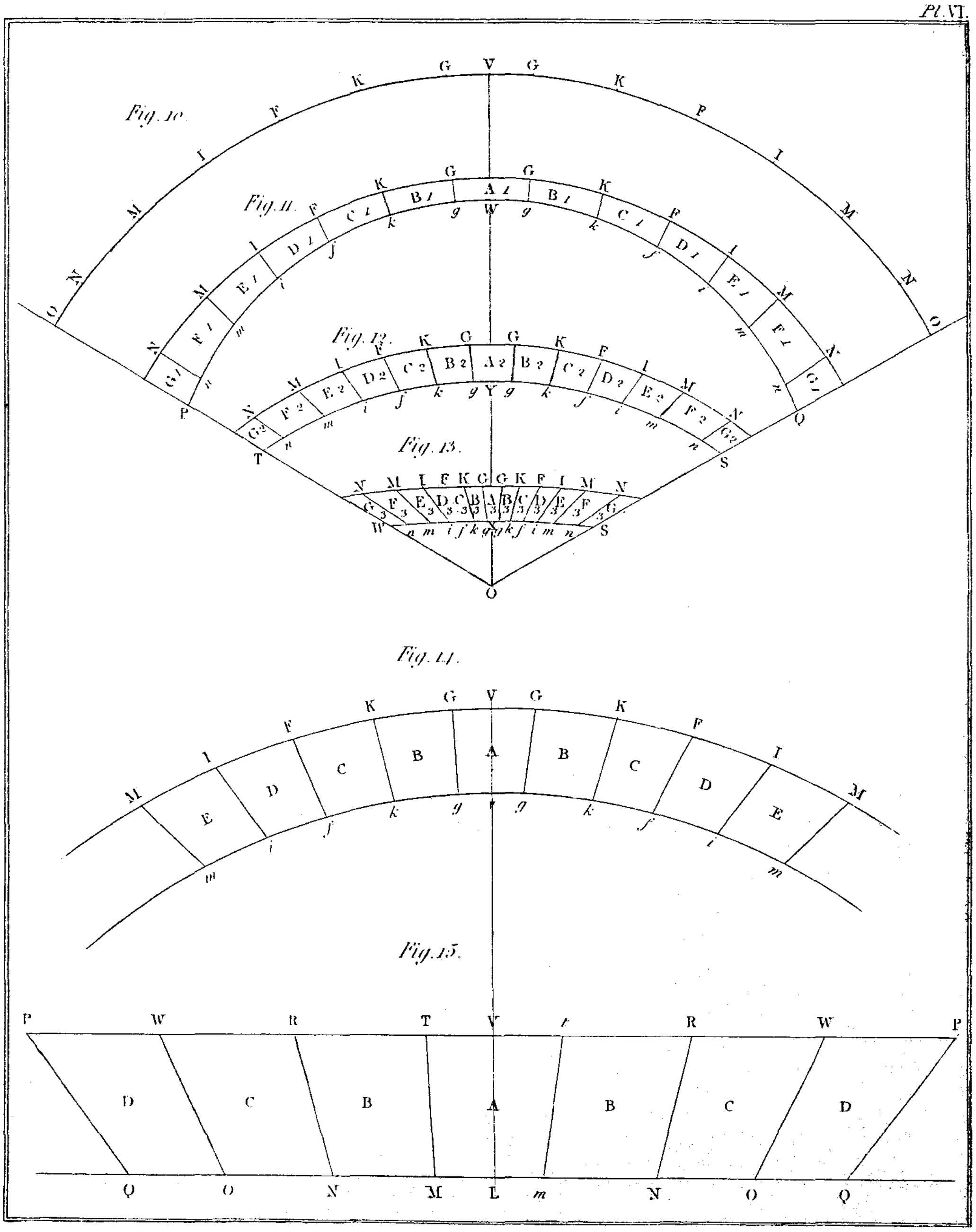



Fig. 9.



Pl. VI.



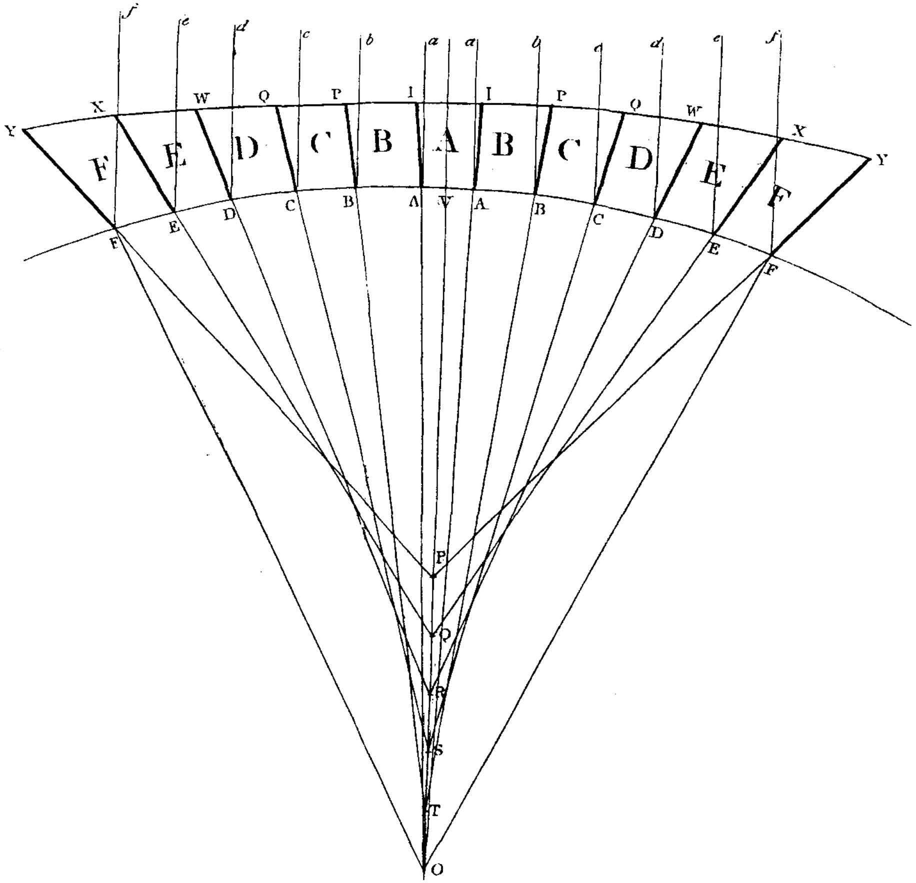

f e d c b a a b c d e f
X W Q P I I P Q W X
Y Y
F E D C B A B C D E F
F E D C B A V A B C D E F
P
Q
R
S
T
O

Fig. 26.